> An Arithmetic for Rigorous Polynomial Approximations

Approximations, Fixed-Point Methods and Algorithms for Function Space Problems

Credit: Galileo / ESA
Power of Floating-Point Scientific Computing...
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Robust IEEE 754 format

Correctly rounded arithmetic operations

Hardware implementation

Hugely optimized Floating-Point Units (FPU) in current processors
Power of Floating-Point Scientific Computing...

- Robust IEEE 754 format
- Correctly rounded arithmetic operations
- Hardware implementation
- Globally optimized Floating-Point Units (FPU) in current processors
- Linear algebra
- Numerical algorithms
- Functional analysis (ODEs, PDEs, ...)

An Arithmetic for Rigorous Polynomial Approximations
...but also Limits and Threats!

numerical errors
...but also Limits and Threats!

**rounding errors**

- each elementary operation
  - induces a small rounding error...

...that accumulates along iterations!

**numerical errors**
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**Rounding errors**

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**Approximation errors**

- Projection methods
- Discretization methods
...but also Limits and Threats!

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- Numerical errors
- Implementation errors
  - Division by zero!
  - Unsafe type casting!
- Approximation errors
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  - Discretization methods
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- Each elementary operation induces a small rounding error...
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- Division by zero!
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- Discretization methods

*Patriot missile failure in 1991 (28 killed)*

**Binary Formats**
- **binary64** ➔ **binary32**
...but also Limits and Threats!

- **Rounding errors**: each elementary operation induces a small rounding error... that accumulates along iterations!
- **Implementation errors**: division by zero! unsafe type casting!
- **Approximation errors**: Patriot missile failure in 1991 (28 killed)
- **Numerical errors**: projection methods

---

**Domains needing rigorous numerics**

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binary64 >> binary32
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domains needing rigorous numerics

safety-critical engineering

computer-assisted mathematics
From Floating-Point Arithmetics to Interval Arithmetics

- **Floating-Point Arithmetic:**

\[ x = (-1)^s \cdot 1.\underbrace{1010011100\ldots}_{52 \text{ bits}} \cdot 2^e \]

**Discretization of the real line:**

- Overapproximation of the reals by intervals:
  - \[ \pi \in [3.14, 3.15] \]
  - \[ e \in [2.71, 2.72] \]

**Interval extension of arithmetic operators:**

- Wrapping effect:
  - Loss of correlation:
    - \[ \cos([0, 2\pi + \epsilon]) - \cos([0, 2\pi]) = [-1, 1] - [-1, 1] = [-2, 2] \]
    - \[ \cos(x + \epsilon) - \cos(x) \leq \epsilon \]
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Overapprox real by intervals

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Interval extension of arithmetic operators:

\[ \pi - e \in [3.14 - 2.72, 3.15 - 2.71] \]
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- Loss of correlation

\[ [-x, x] - [-x, x] = [-2x, 2x] \neq [0, 0]. \]

\[ \cos([0, 2\pi] + \varepsilon) - \cos([0, 2\pi]) = [-1, 1] - [-1, 1] = [-2, 2] \]

but \[ |\cos(x + \varepsilon) - \cos(x)| \leq \varepsilon. \]
Outline

1. Introduction
2. Rigorous Polynomial Approximations
3. A Posteriori Validation with Fixed-Points
4. Validated Solutions of Linear Differential Equations
5. Conclusion and Future Work
6. Some Extras
Outline

1 Introduction

2 Rigorous Polynomial Approximations

3 A Posteriori Validation with Fixed-Points

4 Validated Solutions of Linear Differential Equations

5 Conclusion and Future Work

6 Some Extras

An Arithmetic for Rigorous Polynomial Approximations
a class $\mathcal{F}$ of functions, a reference norm $\| \cdot \|$, and a computable family $\mathcal{P} = (P_n)$ to approximate them.

**Theorem (Stone Weierstrass)**

*The family of polynomials is dense in the set of continuous functions over a compact interval.*
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Taylor expansions...

- monomial basis
- fast computations
- related to initial conditions

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should be used with caution!

- non smooth functions?
- non-analytic functions?
- radius of convergence?
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Chebyshev Polynomials and Series

Chebyshev Family of Polynomials

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\[ T_1(X) = X, \]
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Trigonometric Relation

\[ T_n(\cos \vartheta) = \cos n\vartheta. \]

\[ \forall t \in [-1, 1], \] divides \( T_n(t) \leq 1. \)

Multiplication and Integration

\[ T_n T_m = \frac{1}{2} \left( T_n + m + T_n - m \right). \]

\[ \int T_n = \frac{1}{2} \left( T_n + 1 \right)^n + \frac{1}{2} \left( T_n - 1 \right)^n. \]
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Scalar Product and Orthogonality Relations

\[ \langle f, g \rangle = \int_{-1}^{1} \frac{f(t)g(t)}{\sqrt{1 - t^2}} \, dt = \int_{0}^{\pi} f(\cos \vartheta)g(\cos \vartheta) \, d\vartheta. \]

⇒ \( (T_n)_{n \geq 0} \) orthogonal family.
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**Chebyshev Coefficients and Series**

\[ a_n = \frac{1}{\pi} \int_{0}^{\pi} f(\cos \vartheta) \cos n\vartheta \, d\vartheta, \quad n \in \mathbb{Z}. \]

\[ \hat{f}[N](t) = \sum_{|n| \leq N} a_n T_n(t), \quad t \in [-1, 1]. \]
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**Convergence Theorems**

- If \( f \in C^k, \ \hat{f}[N] \to f \) in \( O(N^{-k}) \).
- If \( f \) analytic, \( \hat{f}[N] \to f \) exponentially fast.
Definition

A pair \((P, \varepsilon) \in \mathbb{R}[X] \times \mathbb{R}_+\) is a rigorous polynomial approximation (RPA) of \(f\) for a given norm \(\| \cdot \|\) if \(\| f - P \| \leq \varepsilon\).
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Example: sup-norm over \([-1, 1]\):

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f \in (P, \varepsilon) \iff |f(t) - P(t)| \leq \varepsilon \quad \forall t \in [-1, 1]
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- \((P, \varepsilon) + (Q, \eta) := (P + Q, \varepsilon + \eta),\)
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- \((P, \varepsilon) - (Q, \eta) := (P - Q, \varepsilon + \eta)\),
- \(\int_0^t (P, \varepsilon) := \left(\int_0^t P(s) \, ds, \varepsilon\right)\) if \(\| f \|_{\infty} = \| f \|_{\infty} \cdot \| g \|_{\infty}\).

Example:

\[r(t) = f(t) + g(t) - h(t)\]
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- \((P, \varepsilon) \cdot (Q, \eta) := (PQ, \| Q \| \eta + \| P \| \varepsilon + \eta \varepsilon)\)

provided that \(\| fg \| \leq \| f \| \| g \|\).
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\[r(t) = \int_0^t k(s)(f(s) + g(s) - h(s))\,ds\]
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Main Idea: A Posteriori Validation

Reformulate the problem as a fixed-point equation $T \cdot x = x$ over metric space $(X, d)$ and obtain $x$ candidate approximation of exact solution $x^*$. 

- Find rigorous error bound $\|x - x^*\|$. 

**Banach Fixed-Point Theorem**

If $(X, d)$ is complete and $T$ contracting of ratio $\mu < 1$, then $T$ admits a unique fixed-point $x^*$, and for all $x \in X$, 

$$d(x, T \cdot x) \leq \frac{1}{1-\mu} d(x, x^*) \leq d(x, T \cdot x) - \mu.$$ 

**Quasi-Newton Method for** $F \cdot x = 0$

Obtain $A \approx (DF)^{-1}$ in order to define: 

$$T \cdot x = x - A \cdot F \cdot x.$$ 

An Arithmetic for Rigorous Polynomial Approximations
Fixed-Point Based Validation
Banach Fixed-Point Theorem

Main Idea: A Posteriori Validation

Reformulate the problem as a fixed-point equation $T \cdot x = x$ over metric space $(X, d)$ and obtain a candidate approximation of exact solution $x^*$. Find rigorous error bound $\|x - x^*\|$.

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If $(X, d)$ is complete and $T$ contracting of ratio $\mu < 1$, then $T$ admits a unique fixed-point $x^*$.
Main Idea: A Posteriori Validation

Reformulate the problem as a fixed-point equation \( T \cdot x = x \) over metric space \((X, d)\) and obtain \( x \) candidate approximation of exact solution \( x^* \).

- Find rigorous error bound \( \|x - x^*\| \).

Banach Fixed-Point Theorem

If \((X, d)\) is complete and \( T \) contracting of ratio \( \mu < 1 \),
- Then \( T \) admits a unique fixed-point \( x^* \),
Main Idea: A Posteriori Validation

Reformulate the problem as a fixed-point equation $T \cdot x = x$ over metric space $(X, d)$ and obtain $x$ candidate approximation of exact solution $x^*$.

- Find **rigorous** error bound $\|x - x^*\|$.

Banach Fixed-Point Theorem

If $(X, d)$ is complete and $T$ **contracting** of ratio $\mu < 1$,

- Then $T$ admits a unique fixed-point $x^*$,
Main Idea: A Posteriori Validation

Reformulate the problem as a fixed-point equation \( T \cdot x = x \) over metric space \((X, d)\)
and obtain \( x \) candidate approximation of exact solution \( x^* \).

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Banach Fixed-Point Theorem

If \((X, d)\) is complete and \( T \) contracting of ratio \( \mu < 1 \),
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Main Idea: A Posteriori Validation

Reformulate the problem as a fixed-point equation \( \mathbf{T} \cdot \mathbf{x} = \mathbf{x} \) over metric space \((X, d)\) and obtain \( \mathbf{x} \) candidate approximation of exact solution \( \mathbf{x}^* \).

- Find rigorous error bound \( \| \mathbf{x} - \mathbf{x}^* \| \).

Banach Fixed-Point Theorem

If \((X, d)\) is complete and \( \mathbf{T} \) contracting of ratio \( \mu < 1 \),

- Then \( \mathbf{T} \) admits a unique fixed-point \( \mathbf{x}^* \),
Main Idea: A Posteriori Validation

Reformulate the problem as a fixed-point equation $T \cdot x = x$ over metric space $(X, d)$ and obtain $x$ candidate approximation of exact solution $x^*$.

- Find rigorous error bound $\|x - x^*\|$.

Banach Fixed-Point Theorem

If $(X, d)$ is complete and $T$ contracting of ratio $\mu < 1$,
- Then $T$ admits a unique fixed-point $x^*$, and
- For all $x \in X$,

$$
\frac{d(x, T \cdot x)}{1 + \mu} \leq d(x, x^*) \leq \frac{d(x, T \cdot x)}{1 - \mu}.
$$
Fixed-Point Based Validation

Banach Fixed-Point Theorem

Main Idea: A Posteriori Validation

Reformulate the problem as a fixed-point equation \( T \cdot x = x \) over metric space \((X, d)\) and obtain \( x \) candidate approximation of exact solution \( x^* \).

- Find rigorous error bound \( \|x - x^*\| \).

Banach Fixed-Point Theorem

If \((X, d)\) is complete and \( T \) contracting of ratio \( \mu < 1 \),

- Then \( T \) admits a unique fixed-point \( x^* \), and
- For all \( x \in X \),

\[
\frac{d(x, T \cdot x)}{1 + \mu} \leq d(x, x^*) \leq \frac{d(x, T \cdot x)}{1 - \mu}.
\]

Quasi-Newton Method for \( F \cdot x = 0 \)

Obtain \( A \approx (DF)^{-1} \) in order to define:

\[
T \cdot x = x - A \cdot F \cdot x.
\]
An Example: Tschauner and Hempel Equation
Relative Motion in Keplerian Dynamics

Reduced Equation

\[ z'' + \left( 4 - \frac{3}{1 + e \cos \nu} \right) z = c \]
An Example: Tschauner and Hempel Equation
Relative Motion in Keplerian Dynamics

Reduced Equation

\[ z'' + \left( 4 - \frac{3}{1 + e \cos \nu} \right) z = c \]

To Do List

1. Approximate coefficient
An Example: Tschauner and Hempel Equation
Relative Motion in Keplerian Dynamics

Reduced Equation

\[ z'' + \left( 4 - \frac{3}{1 + e \cos \nu} \right) z = c \]

To Do List

1. Approximate coefficient
2. Approximate solution with a Chebyshev series
An Example: Tschauner and Hempel Equation
Relative Motion in Keplerian Dynamics

Reduced Equation

\[ z'' + \left( 4 - \frac{3}{1 + e \cos \nu} \right) z = c \]

To Do List

1. Approximate coefficient
2. Approximate solution with a Chebyshev series
3. Validate the obtained solution
An Example: Tschauner and Hempel Equation

Relative Motion in Keplerian Dynamics

Reduced Equation

\[ z'' + \left(4 - \frac{3}{1 + e \cos \nu}\right) z = c \]
An Example: Tschauner and Hempel Equation
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An Example: Tschauner and Hempel Equation
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Reduced Equation

\[ z'' + \left(4 - \frac{3}{1 + e \cos \nu}\right)z = c \]
Approximation of $x \mapsto 4 - \frac{3}{1 + e \cos x}$

✓ RPA for $x \mapsto \cos x$:

$0.77 T_0(x) - 0.23 T_2(x) + 0.005 T_4(x) \pm 4.2 \cdot 10^{-5}$
Fixed-Point Based Validation
Application to Division

**Approximation of** \( x \mapsto 4 - \frac{3}{1 + e \cos x} \)

✓ **RPA for** \( x \mapsto \cos x \):

\[
0.77 T_0(x) - 0.23 T_2(x) + 0.005 T_4(x) \pm 4.2 \cdot 10^{-5}
\]

✓ **RPA for** \( x \mapsto 1 + 0.5 \cos x \):

\[
1.38 T_0(x) - 0.11 T_2(x) + 0.002 T_4(x) \pm 2.1 \cdot 10^{-5}
\]
Fixed-Point Based Validation
Application to Division

Approximation of $x \mapsto 4 - \frac{3}{1 + e \cos x}$

✓ Approximation of $x \mapsto \frac{1}{1 + 0.5 \cos x}$:

![Graph showing the approximation of $1/(1 + 0.5 \cos x)$]
Approximation of \( x \mapsto 4 - \frac{3}{1 + e \cos x} \)

✓ Approximation of \( x \mapsto \frac{1}{1 + 0.5 \cos x} \):

\[
\begin{align*}
\text{Approximation of } x \mapsto 1/(1 + 0.5 \cos x) & : \\
\text{Approximation of } x \mapsto 4 - \frac{3}{1 + e \cos x} & : \\
\end{align*}
\]
Fixed-Point Based Validation
Application to Division

Approximation of \( x \mapsto 4 - \frac{3}{1 + e \cos x} \)

✓ Approximation of \( x \mapsto \frac{1}{1 + 0.5 \cos x} \):

\[
\begin{align*}
T_0(x) &+ T_2(x) \pm 1.3 \cdot 10^{-3} \\
T_0(x) &- 0.18 T_2(x) \pm 3.8 \cdot 10^{-3}
\end{align*}
\]
Approximation of $x \mapsto 4 - \frac{3}{1 + e \cos x}$

✓ Approximation of $x \mapsto 1/(1 + 0.5 \cos x)$:

$\varphi = 0.73 \, T_0(x) + 0.06 \, T_2(x) \approx 1/(1 + 0.5 \cos x)$
Division: \( g/f \) \((f \neq 0)\)

- Solve \( F \cdot \varphi = f \varphi - g = 0 \)

\[
(DF)^{-1} \cdot h = f^{-1} h
\]

\[
(DF) \varphi \cdot h = f h
\]

**Approximation of** \( x \mapsto 4 - \frac{3}{1 + e \cos x} \)

✓ **Approximation of** \( x \mapsto 1/(1 + 0.5 \cos x) \):

\[
\varphi = 0.73 T_0(x) + 0.06 T_2(x) \approx 1/(1 + 0.5 \cos x)
\]
Fixed-Point Based Validation
Application to Division

\[ \frac{g}{f} \quad (f \neq 0) \]

- Solve \( F \cdot \varphi = f \varphi - g = 0 \)

\[ (DF) \varphi \cdot h = fh \quad (DF)^{-1} \cdot h = f^{-1}h \]

- Use \( f_0 \approx f^{-1} \):

\[ T \cdot \varphi = \varphi \quad T \cdot \varphi = \varphi - f_0(f \varphi - g) \]

\[ \varphi = 0.73 T_0(x) + 0.06 T_2(x) \approx 1/(1 + 0.5 \cos x) \]
Fixed-Point Based Validation
Application to Division

Division: \( g/f \ (f \neq 0) \)

- Solve \( F \cdot \varphi = f \varphi - g = 0 \)

\[(DF)^{-1} \cdot h = f^{-1}h\]

- Use \( f_0 \approx f^{-1} \):

\[ T \cdot \varphi = \varphi \quad T \cdot \varphi = \varphi - f_0(f \varphi - g) \]

- \( T \) affine, \( \mu = \|DT\| = \|1 - f_0f\| < 1 \)?

Approximation of \( x \mapsto 4 - \frac{3}{1 + e \cos x} \)

✓ Approximation of \( x \mapsto 1/(1 + 0.5 \cos x) \):

\[ \varphi = 0.73 T_0(x) + 0.06 T_2(x) \approx 1/(1 + 0.5 \cos x) \]
Division: $g/f \ (f \neq 0)$

- Solve $F \cdot \varphi = f\varphi - g = 0$

  \[(DF)\varphi \cdot h = f h \quad (DF)^{-1} \cdot h = f^{-1} h\]

- Use $f_0 \approx f^{-1}$:

  \[T \cdot \varphi = \varphi \quad T \cdot \varphi = \varphi - f_0(f\varphi - g)\]

- $T$ affine, $\mu = \|DT\| = \|1 - f_0 f\| < 1$ ?

Approximation of $x \mapsto 4 - \frac{3}{1 + e \cos x}$

- Approximation of $x \mapsto 1/(1 + 0.5 \cos x)$:

  \[\varphi = 0.73 T_0(x) + 0.06 T_2(x) \approx 1/(1+0.5 \cos x)\]

  - $\mu = 3.7 \cdot 10^{-3} \ll 1$
Fixed-Point Based Validation
Application to Division

**Division: \( g/f \) (\( f \neq 0 \))**

- Solve \( F \cdot \varphi = f \varphi - g = 0 \)

\[
(DF) \varphi \cdot h = fh \quad (DF)^{-1} \cdot h = f^{-1}h
\]

- Use \( f_0 \approx f^{-1} \):

\[
T \cdot \varphi = \varphi \quad T \cdot \varphi = \varphi - f_0(f \varphi - g)
\]

- **\( T \) affine, \( \mu = \|DT\| = \|1 - f_0f\| < 1 \)**?

\[
\|f_0(f \varphi - g)\| \leq \| \varphi - \frac{g}{f} \| \leq \|f_0(f \varphi - g)\| \frac{1}{1 + \mu}
\]

**Approximation of** \( x \mapsto 4 - \frac{3}{1 + e \cos x} \)

\( \varphi = 0.73 T_0(x) + 0.06 T_2(x) \approx 1/(1 + 0.5 \cos x) \)

\( \mu = 3.7 \cdot 10^{-3} \ll 1 \)

\( \checkmark \) RPA for \( x \mapsto 1/(1 + 0.5 \cos x) \):

\[ 0.73 T_0(x) + 0.06 T_2(x) \pm 1.3 \cdot 10^{-3} \]
Fixed-Point Based Validation
Application to Division

Division: \( g/f \) \((f \neq 0)\)

- Solve \( \mathbf{F} \cdot \varphi = f\varphi - g = 0 \)

\[
(\mathbf{DF})\varphi \cdot h = f h \quad (\mathbf{DF})^{-1} \cdot h = f^{-1} h
\]

- Use \( f_0 \approx f^{-1} \): 

\[
T \cdot \varphi = \varphi \quad T \cdot \varphi = \varphi - f_0(f\varphi - g)
\]

- \( T \text{ affine, } \mu = \|\mathbf{DT}\| = \|1 - f_0 f\| < 1 ? \)

\[
\frac{\|f_0(f\varphi - g)\|}{1 + \mu} \leq \|\varphi - \frac{g}{f}\| \leq \frac{\|f_0(f\varphi - g)\|}{1 - \mu}
\]

Approximation of \( x \mapsto 4 - \frac{3}{1 + e \cos x} \)

✓ Approximation of \( x \mapsto 1/(1 + 0.5 \cos x) \):

\[
\varphi = 0.73T_0(x) + 0.06T_2(x) \approx 1/(1+0.5 \cos x)
\]

- \( \mu = 3.7 \cdot 10^{-3} << 1 \)

✓ RPA for \( x \mapsto 1/(1 + 0.5 \cos x) \):

\[
0.73T_0(x) + 0.06T_2(x) \pm 1.3 \cdot 10^{-3}
\]

✓ RPA for \( x \mapsto 4 - 3/(1 + 0.5 \cos x) \):

\[
1.82T_0(x) - 0.18T_2(x) \pm 3.8 \cdot 10^{-3}
\]
LODE and Initial Value Problem

\[ y^{(r)}(t) + \alpha_{r-1}(t)y^{(r-1)}(t) + \cdots + \alpha_1(t)y'(t) + \alpha_0(t)y(t) = g(t) \]  
\[ y(-1) = v_0 \quad y'(-1) = v_1 \quad \ldots \quad y^{(r-1)}(-1) = v_{r-1} \]

\( t \in [-1, 1] \quad \alpha_i, g \) sufficiently regular \((C^0, \text{RPA, polynomial})\)
LODE and Initial Value Problem

\[ y^{(r)}(t) + \alpha_{r-1}(t)y^{(r-1)}(t) + \cdots + \alpha_1(t)y'(t) + \alpha_0(t)y(t) = g(t) \]  \hspace{1cm} (D)

\[ y(-1) = v_0 \quad y'(-1) = v_1 \quad \ldots \quad y^{(r-1)}(-1) = v_{r-1} \]

t \in [-1, 1] \quad \alpha_i, g \text{ sufficiently regular (}C^0, \text{ RPA, polynomial)}

Integral Reformulation

Let \( \varphi = y^{(r)} \), (D) becomes:

\[ \varphi + K \cdot \varphi = \psi, \]  \hspace{1cm} (I)
LODE and Initial Value Problem

\[ y^{(r)}(t) + \alpha_{r-1}(t)y^{(r-1)}(t) + \cdots + \alpha_1(t)y'(t) + \alpha_0(t)y(t) = g(t) \]  \hspace{1cm} (D)

\[ y(-1) = v_0 \quad y'(-1) = v_1 \quad \ldots \quad y^{(r-1)}(-1) = v_{r-1} \]

\[ t \in [-1, 1] \quad \alpha_i, g \text{ sufficiently regular (} C^0, \text{ RPA, polynomial}) \]

Integral Reformulation

Let \( \varphi = y^{(r)} \), (D) becomes:

\[ \varphi + \mathbf{K} \cdot \varphi = \psi, \]  \hspace{1cm} (I)

\[ \mathbf{K} \cdot \varphi(t) = \sum_{j=0}^{r-1} \beta_j(t) \int_{-1}^{t} T_j(s)\varphi(s)ds \]
Linear Ordinary Differential Equations

LODE and Initial Value Problem

\[ y^{(r)}(t) + \alpha_{r-1}(t)y^{(r-1)}(t) + \cdots + \alpha_1(t)y'(t) + \alpha_0(t)y(t) = g(t) \]  

\[ y(-1) = v_0 \quad y'(-1) = v_1 \quad \ldots \quad y^{(r-1)}(-1) = v_{r-1} \]

t \in [-1, 1] \quad \alpha_i, g \text{ sufficiently regular } (C^0, \text{RPA, polynomial})

Integral Reformulation

Let \( \varphi = y^{(r)} \), (D) becomes:

\[ \varphi + \mathbf{K} \cdot \varphi = \psi, \]

\[ \mathbf{K} \cdot \varphi(t) = \sum_{j=0}^{r-1} \beta_j(t) \int_{-1}^{t} T_j(s) \varphi(s) ds \Rightarrow \text{compact operator} \]
**Linear Ordinary Differential Equations**

**LODE and Initial Value Problem**

\[
y^{(r)}(t) + \alpha_{r-1}(t)y^{(r-1)}(t) + \cdots + \alpha_1(t)y'(t) + \alpha_0(t)y(t) = g(t)
\]

\[
y(-1) = v_0 \quad y'(-1) = v_1 \quad \ldots \quad y^{(r-1)}(-1) = v_{r-1}
\]

\[t \in [-1, 1] \quad \alpha_i, g \text{ sufficiently regular } (C^0, \text{RPA, polynomial})\]

**Integral Reformulation**

Let \( \varphi = y^{(r)} \), (D) becomes:

\[
\varphi + \mathbf{K} \cdot \varphi = \psi,
\]

- \( \mathbf{K} \cdot \varphi(t) = \sum_{j=0}^{r-1} \beta_j(t) \int_{-1}^{t} T_j(s) \varphi(s) ds \Rightarrow \text{compact operator} \)
- \( \psi(t) = g(t) + \text{(some function depending on the } v_j \text{'s)} \)
Linear Ordinary Differential Equations

LODE and Initial Value Problem

\[ y^{(r)}(t) + \alpha_{r-1}(t)y^{(r-1)}(t) + \cdots + \alpha_1(t)y'(t) + \alpha_0(t)y(t) = g(t) \]  
\[ y(-1) = v_0 \quad y'(-1) = v_1 \quad \ldots \quad y^{(r-1)}(-1) = v_{r-1} \]

\( t \in [-1, 1] \quad \alpha_i, g \text{ sufficiently regular } (C^0, \text{RPA, polynomial}) \)

Integral Reformulation

Let \( \varphi = y^{(r)} \), (D) becomes:

\[ \varphi + K \cdot \varphi = \psi, \]  
\( (I) \)

- \( K \cdot \varphi(t) = \sum_{j=0}^{r-1} \beta_j(t) \int_{-1}^{t} T_j(s)\varphi(s)ds \Rightarrow \text{compact operator} \)
- \( \psi(t) = g(t) + \text{(some function depending on the } v_j'\text{s)} \)

Theorem (Picard-Lindelöf)

(I) (and hence (D)) has a unique solution.
The Almost-Banded Structure of the Operator $K$

Matrix Representation in Chebyshev Basis

The infinite-dimensional operator $K$. 

An Arithmetic for Rigorous Polynomial Approximations
The finite-dimensional truncation $K^{[N]}$. 

---

The Almost-Banded Structure of the Operator $K$

Matrix Representation in Chebyshev Basis
The Almost-Banded Structure of the Operator $\mathbf{K}$

Example with Tschauner-Hempel Equation

\[
\mathbf{K} \cdot \varphi = t \left( 4 - \frac{3}{1 + e \cos t} \right) \int_{t_0}^{t} \varphi(s) \, ds + \left( -4 + \frac{3}{1 + e \cos t} \right) \int_{t_0}^{t} s \varphi(s) \, ds
\]
The Almost-Banded Structure of the Operator $\mathbf{K}$

Example with Tschauner-Hempel Equation

$$
\mathbf{K} \cdot \varphi \approx t(1.82 - 0.18T_2(t)) \int_{t_0}^{t} \varphi(s)ds + (-1.82 + 0.18T_2(t)) \int_{t_0}^{t} s\varphi(s)ds
$$
The Almost-Banded Structure of the Operator $\mathbf{K}$

Example with Tschauner-Hempel Equation

$$
\mathbf{K} \cdot \varphi \approx (1.73 T_1(t) - 0.09 T_3(t)) \int_{t_0}^{t} \varphi(s) ds + (-1.82 + 0.18 T_2(t)) \int_{t_0}^{t} s \varphi(s) ds
$$
The Almost-Banded Structure of the Operator $K$

Example with Tschauner-Hempel Equation

$$K \cdot \varphi \approx \left(1.73 T_1(t) - 0.09 T_3(t)\right) \int_{t_0}^{t} \varphi(s) ds + \left(-1.82 + 0.18 T_2(t)\right) \int_{t_0}^{t} s \varphi(s) ds$$

An Arithmetic for Rigorous Polynomial Approximations

12/17
We want to solve $z''(t) + \left( 4 - \frac{3}{1 + 0.5 \cos t} \right) z(t) = c$ with $z(-1) = 0$, $z'(-1) = 1$ and $c = 1$.
We want to solve $z''(t) + \left(4 - \frac{3}{1 + 0.5\cos t}\right)z(t) = c$ with $z(-1) = 0$, $z'(-1) = 1$ and $c = 1$.

Equivalent to $(I + K) \cdot \varphi = \psi$ where $\varphi = z''$. 

An Arithmetic for Rigorous Polynomial Approximations
We want to solve \( z''(t) + \left(4 - \frac{3}{1 + 0.5 \cos t}\right) z(t) = c \) with \( z(-1) = 0 \), \( z'(-1) = 1 \) and \( c = 1 \).

\( \approx \) Equivalent to \( (I + K^{[N]}) \cdot \varphi = \psi \) where \( \varphi = z'' \).
Approximate Solution to Tschauner-Hemple Equation

- We want to solve 
  \[ z''(t) + \left(4 - \frac{3}{1 + 0.5 \cos t}\right) z(t) = c \] 
  with \( z(-1) = 0, z'(-1) = 1 \) and \( c = 1 \).
- \( \approx \) Equivalent to 
  \[ \left( I + K^N \right) \cdot \varphi = \psi \] 
  where \( \varphi = z'' \).
- We have a matrix representation of \( I + K^N \).
We want to solve \( z''(t) + \left( 4 - \frac{3}{1 + 0.5 \cos t} \right) z(t) = c \) with \( z(-1) = 0, \ z'(-1) = 1 \) and \( c = 1 \).

\[ \approx \text{Equivalent to } (I + K^{[N]}) \cdot \varphi = \psi \text{ where } \varphi = z'' . \]

We have a matrix representation of \( I + K^{[N]} \).

\[ \psi \approx -0.82T_0 - 1.73T_1 + 0.18T_2 + 0.09T_3. \]
We want to solve $z''(t) + \left(4 - \frac{3}{1 + 0.5 \cos t}\right) z(t) = c$ with $z(-1) = 0$, $z'(-1) = 1$ and $c = 1$.

\[ \approx \text{Equivalent to } \left( I + K^\text{[N]} \right) \cdot \varphi = \psi \text{ where } \varphi = z''. \]

We have a matrix representation of $I + K^\text{[N]}$.

\[ \psi \approx -0.82 T_0 - 1.73 T_1 + 0.18 T_2 + 0.09 T_3. \]

Hence, by inverting the linear system, we get:
We want to solve \( z''(t) + \left(4 - \frac{3}{1+0.5 \cos t}\right) z(t) = c \) with \( z(-1) = 0, \ z'(-1) = 1 \) and \( c = 1 \).

\( \approx \) Equivalent to \( (\mathbf{I} + \mathbf{K}^N) \varphi = \psi \) where \( \varphi = z'' \).

We have a matrix representation of \( \mathbf{I} + \mathbf{K}^N \).

\( \psi \approx -0.82 T_0 - 1.73 T_1 + 0.18 T_2 + 0.09 T_3. \)

Hence, by inverting the linear system, we get:

\[
\varphi = -0.6 T_0 - 1.19 T_1 + 0.62 T_2 + 0.17 T_3 - 0.05 T_4 - 0.01 T_5 \\
+ 2.1 \cdot 10^{-3} T_6 + 3.2 \cdot 10^{-3} T_7 - 5.8 \cdot 10^{-5} T_8 - 7.6 \cdot 10^{-6} T_9 + 1.2 \cdot 10^{-6} T_{10} \\
+ 1.4 \cdot 10^{-7} T_{11} - 1.9 \cdot 10^{-8} T_{12} - 2.0 \cdot 10^{-9} T_{13} + 2.6 \cdot 10^{-10} T_{14} + 2.5 \cdot 10^{-11} T_{15} \\
- 3.0 \cdot 10^{-12} T_{16} - 2.6 \cdot 10^{-13} T_{17} + 3.0 \cdot 10^{-14} T_{18} + 2.5 \cdot 10^{-15} T_{19} - 2.6 \cdot 10^{-16} T_{20}
\]
Designing the Newton-like Operator $T$

**Construct $T$: To-Do List**

- Truncation order $N$. 

Approx inverse:

$$A \approx (I + K)^{-1}$$

Decomposition of the Operator Norm

$$\|DT\|_{\text{alt1}} = \|I - A(I + K)^{-1}\|_{\text{alt1}} \leq \|I - A(I + K)\|_{\text{alt1}} + \|A(K - K[N])\|_{\text{alt1}}.$$ 

Approximation error:

Truncation error:

Involves basic arithmetic operations on matrices: multiplication, addition, 1-norm.

Determines the minimal value of $N$ we can choose.
Construct $T$: To-Do List

- Truncation order $N$.
- Approx inverse:

\[ A \approx (I + K)^{-1} \]
Construct $T$: To-Do List

- Truncation order $N$.
- Approx inverse:

$$A \approx (I + K^N)^{-1}$$
Designing the Newton-like Operator $T$

Construct $T$: To-Do List

- Truncation order $N$.
- Approx inverse:

$$A \approx (I + K^N)^{-1}$$

Decomposition of the Operator Norm

$$\|DT\| = \|I - A(I+K)\| \leq \|I - A(I+K^N)\| + \|A(K - K^N)\|.$$
Designing the Newton-like Operator $T$

**Construct $T$: To-Do List**

- Truncation order $N$.
- Approx inverse:
  \[ A \approx (I + K^N)^{-1} \]

**Decomposition of the Operator Norm**

\[
\|DT\| = \|I - A(I+K)\| \leq \|I - A(I + K^N)\| + \|A(K - K^N)\|. 
\]

Approximation error
Construct T: To-Do List

- Truncation order \( N \).
- Approx inverse:

\[
A \approx (I + K^{[N]})^{-1}
\]

Decomposition of the Operator Norm

\[
\|DT\| = \|I - A(I + K)\| \leq \|I - A(I + K^{[N]})\| + \|A(K - K^{[N]})\|.
\]

Approximation error

Truncation error
Designing the Newton-like Operator \( T \)

**Construct \( T \): To-Do List**

- Truncation order \( N \).
- Approx inverse:

\[
A \approx (I + K^{[N]})^{-1}
\]

**Decomposition of the Operator Norm**

\[
\|DT\| = \|I - A(I + K)\| \leq \|I - A(I + K^{[N]})\| + \|A(K - K^{[N]})\|.
\]

**Approximation error:**

- Involves basic arithmetic operations on matrices:
  - Multiplication
  - Addition
  - 1-norm
Construct T: To-Do List

- Truncation order $N$.
- Approx inverse:

$$A \approx (I + K^{[N]})^{-1}$$

Approximation error:

- Involves basic arithmetic operations on matrices:
  - Multiplication
  - Addition
  - 1-norm

Decomposition of the Operator Norm

$$\|DT\| = \|I - A(I + K)\| \leq \|I - A(I + K^{[N]})\| + \|A(K - K^{[N]})\|.$$  

Truncation error:

- Determines the minimal value of $N$ we can choose.
\[ L \cdot y = y^{(r)}(t) + \alpha_{r-1}(t)y^{(r-1)}(t) + \cdots + \alpha_1(t)y'(t) + \alpha_0(t)y(t) = g(t) \]
\[ y(-1) = v_0 \quad y'(-1) = v_1 \quad \ldots \quad y^{(r-1)}(-1) = v_{r-1} \]

Rigorous Solving - Overview

1. Integral reformulation: \( \varphi + K \cdot \varphi = \psi \) with \( \varphi = y^{(r)} \),
2. Numerical solving: approximation \( \varphi \) of \( \varphi^* \),
3. Creating Newton-like operator: \( T \cdot \varphi = \varphi \),
4. Obtaining \( \mu \geq \|DT\| \),
5. If \( \mu < 1 \), \( \|\varphi - \varphi^*\| \leq \varepsilon := \|\varphi - T \cdot \varphi\|/(1 - \mu) \),
6. Integrate RPA \((\varphi, \varepsilon)\) \( r \) times with initial conditions to obtain a RPA for \( y^* \).
Integration of LODEs in RPA Arithmetics

\[ \mathbf{L} \cdot y = y^{(r)}(t) + \alpha_{r-1}(t)y^{(r-1)}(t) + \cdots + \alpha_1(t)y'(t) + \alpha_0(t)y(t) = g(t) \]

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- Extension of the method to RPA coefficients \( \alpha_i = (\tilde{\alpha}_i, \varepsilon_i) \)
Integration of LODEs in RPA Arithmetics

\[ \mathbf{L} \cdot y = y^{(r)}(t) + \alpha_{r-1}(t)y^{(r-1)}(t) + \cdots + \alpha_1(t)y'(t) + \alpha_0(t)y(t) = g(t) \]

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Rigorous Solving - Overview

1. Integral reformulation: \( \varphi + \mathbf{K} \cdot \varphi = \psi \) with \( \varphi = y^{(r)} \),
2. Numerical solving: approximation \( \varphi \) of \( \varphi^* \),
3. Creating Newton-like operator: \( \mathbf{T} \cdot \varphi = \varphi \),
4. Obtaining \( \mu \geq \| \mathbf{DT} \| \),
5. If \( \mu < 1 \), \( \| \varphi - \varphi^* \| \leq \varepsilon := \| \varphi - \mathbf{T} \cdot \varphi \|/(1 - \mu) \),
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- Extension of the method to RPA coefficients \( \alpha_i = (\tilde{\alpha}_i, \varepsilon_i) \)
- Extension to Boundary Value Problems (BVP)
Solution to Tschauner and Hempel Equations

Bring our Example to the End

- Approximation error $\leq 1.5 \cdot 10^{-3}$. 
Solution to Tschauner and Hempel Equations

Bring our Example to the End

- Approximation error $\leq 1.5 \cdot 10^{-3}$.

- Truncation error $\leq 1.21 \cdot 10^{-2}$.

An Arithmetic for Rigorous Polynomial Approximations
Solution to Tschauner and Hempel Equations
Bring our Example to the End

- Approximation error $\leq 1.5 \cdot 10^{-3}$.
- Truncation error $\leq 1.21 \cdot 10^{-2}$.
- $\mu \leq 1.5 \cdot 10^{-3} + 1.21 \cdot 10^{-2}$. 

Hence:

$$\left| T \cdot \phi - \phi \right| \leq A \left( \phi + K \cdot \phi - \psi \right) = 6.48 \cdot 10^{-16}.$$
Solution to Tschauner and Hempel Equations

Bring our Example to the End

- Approximation error $\leq 1.5 \cdot 10^{-3}$.

- Truncation error $\leq 1.21 \cdot 10^{-2}$.

- $\mu \leq 1.5 \cdot 10^{-3} + 1.21 \cdot 10^{-2} = 1.36 \cdot 10^{-2}$. 

Take into account approximation error of coefficient!
Approximation error $\leq 1.5 \cdot 10^{-3}$.

Truncation error $\leq 1.21 \cdot 10^{-2}$.

$\mu \leq 1.5 \cdot 10^{-3} + 1.21 \cdot 10^{-2} = 1.36 \cdot 10^{-2}$.

$\| T \cdot \varphi - \varphi \| = \| A(\varphi + K \cdot \varphi - \psi) \| = 6.48 \cdot 10^{-16}$. 
Solution to Tschauner and Hempel Equations

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- Approximation error $\leq 1.5 \cdot 10^{-3}$.

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- $\mu \leq 1.5 \cdot 10^{-3} + 1.21 \cdot 10^{-2} = 1.36 \cdot 10^{-2}$.

- $\|T \cdot \varphi - \varphi\| = \|A(\varphi + K \cdot \varphi - \psi)\| = 6.48 \cdot 10^{-16}$.

- Hence:

$$\frac{6.48 \cdot 10^{-16}}{1 + \mu} \leq \|\varphi - \varphi^*\| \leq \frac{6.48 \cdot 10^{-16}}{1 - \mu}$$
Solution to Tschauner and Hempel Equations

Bring our Example to the End

- Approximation error $\leq 1.5 \cdot 10^{-3}$.

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- $\mu \leq 1.5 \cdot 10^{-3} + 1.21 \cdot 10^{-2} = 1.36 \cdot 10^{-2}$.

- $\| \mathbf{T} \cdot \varphi - \varphi \| = \| A(\varphi + \mathbf{K} \cdot \varphi - \psi) \| = 6.48 \cdot 10^{-16}$.

- Hence:

  $6.39 \cdot 10^{-16} \leq \| \varphi - \varphi^* \| \leq 6.57 \cdot 10^{-16}$
Approximation error $\leq 1.5 \cdot 10^{-3}$.

Truncation error $\leq 1.21 \cdot 10^{-2}$.

$\mu \leq 1.5 \cdot 10^{-3} + 1.21 \cdot 10^{-2} = 1.36 \cdot 10^{-2}$.

$\| T \cdot \varphi - \varphi \| = \| A(\varphi + K \cdot \varphi - \psi) \| = 6.48 \cdot 10^{-16}$.

Hence:

$$6.39 \cdot 10^{-16} \leq \| \varphi - \varphi^* \| \leq 6.57 \cdot 10^{-16}$$

⚠️ Take into account approximation error of coefficient!
A general framework for an arithmetic of RPAs in Chebyshev basis.
A general framework for an arithmetic of RPAs in Chebyshev basis.

An efficient algorithm to compute RPAs for LODEs:
- Coefficients represented by RPAs.
- Extension to the vectorial case + a new fixed-point theorem for vector-valued problems.

Future directions:
- Non-linear ODEs.
- Other orthogonal families of polynomials.
- A Coq implementation.
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> An Arithmetic for Rigorous Polynomial Approximations

Approximations, Fixed-Point Methods and Algorithms for Function Space Problems

Credit: Galileo / ESA
Outline

1. Introduction

2. Rigorous Polynomial Approximations

3. A Posteriori Validation with Fixed-Points

4. Validated Solutions of Linear Differential Equations

5. Conclusion and Future Work

6. Some Extras
Division: $g/f$ ($f \neq 0$)

- Solve $F \cdot \varphi = f \varphi - g = 0$

\[
(DF')\varphi \cdot h = fh \\
(DF)^{-1} \cdot h = f^{-1} h
\]
Division: \( g/f \ (f \neq 0) \)

- Solve \( F \cdot \varphi = f \varphi - g = 0 \)

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(DF') \varphi \cdot h = fh \quad (DF)^{-1} \varphi \cdot h = f^{-1}h
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- Use \( f_0 \approx f^{-1} \):

\[
T \cdot \varphi = \varphi \quad T \cdot \varphi = \varphi - f_0(f \varphi - g)
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- \( T \text{ affine}, \mu = \|DT\| = \|1 - f_0 f\| < 1 \)?
Fixed-Point Based Validation
Application to Division and Square Root

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T \cdot \varphi = \varphi \quad T \cdot \varphi = \varphi - f_0(f\varphi - g)
\]

- **T affine**, \( \mu = \|DT\| = \|1 - f_0f\| < 1 \):

\[
\frac{\|f_0(f\varphi - g)\|}{1 + \mu} \leq \left\| \varphi - \frac{g}{f} \right\| \leq \frac{\|f_0(f\varphi - g)\|}{1 - \mu}
\]
Fixed-Point Based Validation
Application to Division and Square Root

### Division: $g/f$ ($f \neq 0$)

- Solve $F \cdot \varphi = f \varphi - g = 0$

  $$(DF) \cdot h = fh \quad (DF)^{-1} \cdot h = f^{-1}h$$

- Use $f_0 \approx f^{-1}$:

  $T \cdot \varphi = \varphi \quad T \cdot \varphi = \varphi - f_0(f \varphi - g)$

- $T$ affine, $\mu = \|DT\| = \|1 - f_0f\| < 1$?

  $$\frac{f_0(f \varphi - g)}{1 + \mu} \leq \|\varphi - \frac{g}{f}\| \leq \frac{f_0(f \varphi - g)}{1 - \mu}$$

### Square Root: $\sqrt{f}$ ($f > 0$)

- Solve $F \cdot \varphi = \varphi^2 - f = 0$  \[\text{⚠ Two solutions!}\]

  $$(DF) \cdot h = 2\varphi h \quad (DF)^{-1} \cdot h = \frac{\varphi^{-1}}{2}h$$

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An Arithmetic for Rigorous Polynomial Approximations
Fixed-Point Based Validation
Application to Division and Square Root

Division: $g/f \ (f \neq 0)$

- Solve $F \cdot \varphi = f \varphi - g = 0$
  
  $$(DF)\varphi \cdot h = fh \quad (DF)^{-1} \cdot h = f^{-1}h$$

- Use $f_0 \approx f^{-1}$:
  
  $$T \cdot \varphi = \varphi \quad T \cdot \varphi = \varphi - f_0(f \varphi - g)$$

- $T$ affine, $\mu = \|DT\| = \|1 - f_0 f\| < 1$?

$$\frac{\|f_0(f \varphi - g)\|}{1 + \mu} \leq \|\varphi - \frac{g}{f}\| \leq \frac{\|f_0(f \varphi - g)\|}{1 - \mu}$$

Square Root: $\sqrt{f} \ (f > 0)$

- Solve $F \cdot \varphi = \varphi^2 - f = 0$  \(\text{⚠ Two solutions!}\)
  
  $$(DF)\varphi \cdot h = 2\varphi h \quad (DF)^{-1} \cdot h = \frac{\varphi^{-1}}{2} h$$

- Use $f_0 \approx \varphi^{-1} \ (\approx 1/\sqrt{f})$:
  
  $$T \cdot \varphi = \varphi \quad T \cdot \varphi = \varphi - \frac{f_0}{2}(\varphi^2 - f)$$
Division: \( \frac{g}{f} \) (\( f \neq 0 \))

- Solve \( \mathbf{F} \cdot \varphi = f \varphi - g = 0 \)
  
  \[
  (\mathbf{DF})_{\varphi} \cdot h = fh \quad (\mathbf{DF})_{\varphi}^{-1} \cdot h = f^{-1}h
  \]

- Use \( f_0 \approx f^{-1} \):
  
  \[
  \mathbf{T} \cdot \varphi = \varphi \quad \mathbf{T} \cdot \varphi = \varphi - f_0(f \varphi - g)
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- \( \mathbf{T} \) affine, \( \mu = \| \mathbf{D} \mathbf{T} \| = \| 1 - f_0 f \| < 1 \)?

  \[
  \frac{\| f_0(f \varphi - g) \|}{1 + \mu} \leq \| \varphi - \frac{g}{f} \| \leq \frac{\| f_0(f \varphi - g) \|}{1 - \mu}
  \]

Square Root: \( \sqrt{f} \) (\( f > 0 \))

- Solve \( \mathbf{F} \cdot \varphi = \varphi^2 - f = 0 \) \( \text{⚠️ Two solutions!} \)
  
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  (\mathbf{DF})_{\varphi} \cdot h = 2\varphi h \quad (\mathbf{DF})_{\varphi}^{-1} \cdot h = \frac{\varphi^{-1}}{2}h
  \]

- Use \( f_0 \approx \varphi^{-1} \) (\( \approx 1/\sqrt{f} \)):
  
  \[
  \mathbf{T} \cdot \varphi = \varphi \quad \mathbf{T} \cdot \varphi = \varphi - \frac{f_0}{2} (\varphi^2 - f)
  \]

- \( \mathbf{T} \) non-linear, \( \| (\mathbf{D} \mathbf{T})_{\varphi} \| = \| 1 - f_0 \varphi \|. \)
**Fixed-Point Based Validation**

**Application to Division and Square Root**

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**Division: \( g/f \) (\( f \neq 0 \))**

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- **T non-linear, \( \|DT\varphi\| = \|1 - f_0 \varphi\| \).**

  \( \exists r > 0 \cdot \ T : \overline{B}(\varphi, r) \rightarrow \overline{B}(\varphi, r) \) ?

  \[
  \|T \cdot \varphi - \varphi\| + \sup_{\psi \in \overline{B}(\varphi, r)} \|DT\psi\| < r
  \]

---

`dessin inclusion boule image`
Fixed-Point Based Validation
Application to Division and Square Root

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- $T$ non-linear, $\|DT\varphi\| = \|1 - f_0 \varphi\|$.

  $\exists r > 0 \cdot T : \overline{B}(\varphi, r) \rightarrow \overline{B}(\varphi, r)$?

  $$\|T \cdot \varphi - \varphi\| + r \sup_{\psi \in \overline{B}(\varphi, r)} \|DT\psi\| < r$$

- Check $\Delta \geq 0 \rightarrow 0 \leq r_{\text{min}} \leq r_{\text{max}}$.
- Check $\mu = \|1 - f_0 \varphi\| + \|f_0\| r_{\text{min}} < 1$. 

dessin inclusion boule image
Division: $g/f$ ($f \neq 0$)

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dessin inclusion boule image

Square Root: $\sqrt{f}$ ($f > 0$)

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\exists r > 0 \cdot T : \overline{B}(\varphi, r) \rightarrow \overline{B}(\varphi, r) \Rightarrow \\
\|T \cdot \varphi - \varphi\| + r \sup_{\psi \in \overline{B}(\varphi, r)} \|(DT)\psi\| < r
\]

- Check $\Delta \geq 0$ \(\rightarrow\) $0 \leq r_{\text{min}} \leq r_{\text{max}}$

- Check $\mu = \|1 - f_0 \varphi\| + \|f_0\| r_{\text{min}} < 1$

\[
\frac{f_0 (\varphi^2 - f)/2}{1 + \mu} \leq \varphi - \sqrt{f} \leq \frac{f_0 (\varphi^2 - f)/2}{1 - \mu}
\]
Designing the Newton-like Operator $T$

Bounding the Truncation Error

**Truncation Error**

$$\|A \cdot (K - K^N)\| = \sup_{i \geq 0} \|A \cdot (K - K^N) \cdot T_i\|$$
Designing the Newton-like Operator $\mathbf{T}$

Bounding the Truncation Error

**Truncation Error**

$$\| \mathbf{A} \cdot (\mathbf{K} - \mathbf{K}^N) \| = \sup_{i \geq 0} \| \mathbf{A} \cdot (\mathbf{K} - \mathbf{K}^N) \cdot T_i \|$$
Designing the Newton-like Operator $T$

Bounding the Truncation Error

\[ \| \mathbf{A} \cdot (\mathbf{K} - \mathbf{K}^[[N]]) \| = \sup_{i \geq 0} \| \mathbf{A} \cdot (\mathbf{K} - \mathbf{K}^[[N]]) \cdot T_i \| \]
Designing the Newton-like Operator $T$

Bounding the Truncation Error

The Truncation Error is given by:
$$\|A \cdot (K - K^{[N]})\| = \sup_{i \geq 0} \|A \cdot (K - K^{[N]}) \cdot T_i\|$$

Diagram illustrating the matrices $A$ and $K - K^{[N]}$. The diagram shows the relationship between the matrices, with $A$ on the left and $K - K^{[N]}$ on the right, highlighting the truncation error in the context of polynomial approximations.
Designing the Newton-like Operator $T$
Bounding the Truncation Error

\[ \| A \cdot (K - K^N) \| = \sup_{i \geq 0} \left\| A \cdot (K - K^N) \cdot T_i \right\| \]
Designing the Newton-like Operator $T$

Bounding the Truncation Error

**Truncation Error**

$$\|A \cdot (K - K^{[N]})\| = \sup_{i \geq 0} \|A \cdot (K - K^{[N]}) \cdot T_i\|$$

1. Direct computation.
Designing the Newton-like Operator $T$

Bounding the Truncation Error

**Truncation Error**

$$\| \mathbf{A} \cdot (\mathbf{K} - \mathbf{K}^{[N]}) \| = \sup_{i \geq 0} \| \mathbf{A} \cdot (\mathbf{K} - \mathbf{K}^{[N]}) \cdot T_i \|$$

1. Direct computation.
2. Direct computation.

An Arithmetic for Rigorous Polynomial Approximations
Designing the Newton-like Operator $\mathbf{T}$

Bounding the Truncation Error

\[ \| \mathbf{A} \cdot (\mathbf{K} - \mathbf{K}^N) \| = \sup_{i \geq 0} \| \mathbf{A} \cdot (\mathbf{K} - \mathbf{K}^N) \cdot T_i \| \]

\[ \begin{array}{cccccc}
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 \\
\end{array} \]

\[ \begin{array}{cccccc}
1 & 1 & 1 & 1 & 1 & 1 \\
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1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 \\
\end{array} \]

1. Direct computation.
2. Direct computation.
3. Bound the remaining infinite number of columns:

\[ \text{Cost: } O(\mathbf{N}(\mathbf{h} + \mathbf{d})) \text{ or } O((\mathbf{h}'+\mathbf{d}')(\mathbf{h}+\mathbf{d}')) \]
Designing the Newton-like Operator \( T \)
Bounding the Truncation Error

**Truncation Error**

\[
\| A \cdot (K - K^{[N]}) \| = \sup_{i \geq 0} \| A \cdot (K - K^{[N]}) \cdot T_i \|
\]

1. Direct computation.
2. Direct computation.
3. Bound the remaining *infinite* number of columns:
   - Using the bounds in \( 1/i \) and \( 1/i^2 \): possibly large overestimations.
   
   \[
   \text{diag}(i) \leq \frac{C}{i} \quad \text{init}(i) \leq \frac{D}{i^2}
   \]
Designing the Newton-like Operator $T$

Bounding the Truncation Error

$$\|A \cdot (K - K^{[N]})\| = \sup_{i \geq 0} \|A \cdot (K - K^{[N]}) \cdot T_i\|$$

1. Direct computation.
2. Direct computation.
3. Bound the remaining *infinite* number of columns:
   - Using the bounds in $1/i$ and $1/i^2$: possibly large overestimations.
     $$\text{diag}(i) \leq \frac{C}{i} \quad \text{init}(i) \leq \frac{D}{i^2}$$
   - Using a first order difference method: differences in $1/i^2$ and $1/i^4$.
     $$\text{diag}(i) \leq \text{diag}(i_0) + \frac{C'}{i^2}$$
     $$\text{init}(i) \leq \text{init}(i_0) + \frac{D'}{i^4}$$
Designing the Newton-like Operator $\mathbf{T}$

Bounding the Truncation Error

\[
\| \mathbf{A} \cdot (\mathbf{K} - \mathbf{K}^{[N]}) \| = \sup_{i \geq 0} \| \mathbf{A} \cdot (\mathbf{K} - \mathbf{K}^{[N]}) \cdot T_i \|
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     \]
   - Using a first order difference method: differences in $1/i^2$ and $1/i^4$.
     \[
     \text{diag}(i) \leq \text{diag}(i_0) + \frac{C'}{i^2} \quad \text{init}(i) \leq \text{init}(i_0) + \frac{D'}{i^4}
     \]

Cost: $O(N(h + d))$ or $O((h' + d')(h + d))$

An Arithmetic for Rigorous Polynomial Approximations
Coupled Systems of Linear Ordinary Differential Equations

Coupled LODEs and Initial Value Problem

\[ Y^{(r)}(t) + A_{r-1}(t) \cdot Y^{(r-1)}(t) + \cdots + A_1(t) \cdot Y'(t) + A_0(t) \cdot Y(t) = G(t) \quad (p-D) \]

\[
A_k(t) = \begin{pmatrix}
    a_{k11}(t) & \cdots & a_{k1p}(t) \\
    \vdots & \ddots & \vdots \\
    a_{kp1}(t) & \cdots & a_{kpp}(t)
\end{pmatrix} 
\quad G(t) = \begin{pmatrix}
    g_1(t) \\
    \vdots \\
    g_p(t)
\end{pmatrix}
\]

\[ t \in [-1, 1] \quad Y^{(k)}_i(-1) = v_{ik} \quad i \in [1, p], k \in [0, r - 1] \]
Coupled LODEs and Initial Value Problem

\[ Y^{(r)}(t) + A_{r-1}(t) \cdot Y^{(r-1)}(t) + \cdots + A_1(t) \cdot Y'(t) + A_0(t) \cdot Y(t) = G(t) \quad (p-D) \]

\[ A_k(t) = \begin{pmatrix}
  a_{k11}(t) & \cdots & a_{k1p}(t) \\
  \vdots & \ddots & \vdots \\
  a_{kp1}(t) & \cdots & a_{kpp}(t)
\end{pmatrix} \quad G(t) = \begin{pmatrix}
  g_1(t) \\
  \vdots \\
  g_p(t)
\end{pmatrix} \]

\[ t \in [-1, 1] \quad Y_i^{(k)}(-1) = v_{ik} \quad i \in [1, p], k \in [0, r - 1] \]

Integral Reformulation

Posing \( \Phi = Y^{(r)} \), System (p-D) is transformed into:

\[ \Phi(t) + \int_{t_0}^{t} \begin{pmatrix}
  k_{11}(t, s) & \cdots & k_{1p}(t, s) \\
  \vdots & \ddots & \vdots \\
  k_{p1}(t) & \cdots & k_{pp}(t)
\end{pmatrix} \cdot \Phi(s) ds = \Psi(t) \quad (p-l) \]
The Almost-Banded Structure of the Operator $\mathbf{K}$

Example in Dimension 4

$$
\begin{array}{cccc}
K_{1,1} & K_{1,2} & K_{1,3} & K_{1,4} \\
K_{2,1} & K_{2,2} & K_{2,3} & K_{2,4} \\
K_{3,1} & K_{3,2} & K_{3,3} & K_{3,4} \\
K_{4,1} & K_{4,2} & K_{4,3} & K_{4,4} \\
\end{array}
$$
The Almost-Banded Structure of the Operator $\mathbf{K}$

Example in Dimension 4

\[
\begin{array}{cccc}
  \text{K} & \text{K} & \text{K} & \text{K} \\
  \text{K} & \text{K} & \text{K} & \text{K} \\
  \text{K} & \text{K} & \text{K} & \text{K} \\
  \text{K} & \text{K} & \text{K} & \text{K} \\
\end{array}
\]
The Almost-Banded Structure of the Operator $K$

Example in Dimension 4

$K^N$
The Almost-Banded Structure of the Operator $K$

Example in Dimension 4
The Almost-Banded Structure of the Operator $\mathbf{K}$

Example in Dimension 4

$\mathbf{K}^[[N]]$ (rearranged basis)
## Vector-Valued Metric and Contractions

Let $(X_1, d_1), \ldots, (X_p, d_p)$ be complete metric spaces.

- $d(x, y) = (d_1(x_1, y_1), \ldots, d_p(x_p, y_p)) \in \mathbb{R}^p$ is a vector-valued metric.

- $f : X \to X$ is $\Lambda$-Lipschitz for $\Lambda \in \mathbb{R}^{p \times p}_+$ iff:

  $$d(f(x), f(y)) \leq \Lambda \cdot d(x, y) \quad \forall x, y \in X$$

- $f : X \to X$ is a contraction if it is $\Lambda$-Lipschitz for $\Lambda \in \mathbb{R}^{p \times p}_+$ such that $\Lambda^k \to 0$ as $k \to \infty$. 

---

**Error Polytope**

Let $\varepsilon = d(x, x^*)$ and $\eta = d(x, T \cdot x)$:

$$\begin{align*}
(1 - \Lambda) \cdot \varepsilon &\leq \eta \\
(1 + \Lambda) \cdot \varepsilon &\geq \eta
\end{align*}$$
Vector-Valued Metric and Contractions

\[(X_1, d_1), \ldots, (X_p, d_p)\) complete metric spaces.

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- \(f : X \to X\) is a contraction if it is \(\Lambda\)-Lipschitz for \(\Lambda \in \mathbb{R}^{p \times p}_+\) s.t. \(\Lambda^k \to 0\) as \(k \to \infty\).

**Perov:** \(T\) admits a unique fixed-point \(x^*\).
Vector-Valued Fixed-Point Validation

Vector-Valued Metric and Contractions

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**Perov:** \(T\) admits a unique fixed-point \(x^*\).

- \(d(x, x^*) \leq d(x, T \cdot x) + d(T \cdot x, x^*)\).
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Vector-Valued Fixed-Point Validation

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**Error Polytope**

Let \(\varepsilon = d(x, x^*)\) and \(\eta = d(x, T \cdot x)\):

\[
(1 - \Lambda) \cdot \varepsilon \leq \eta \quad (P)
\]

\[
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\]

\[
\varepsilon \geq 0
\]
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