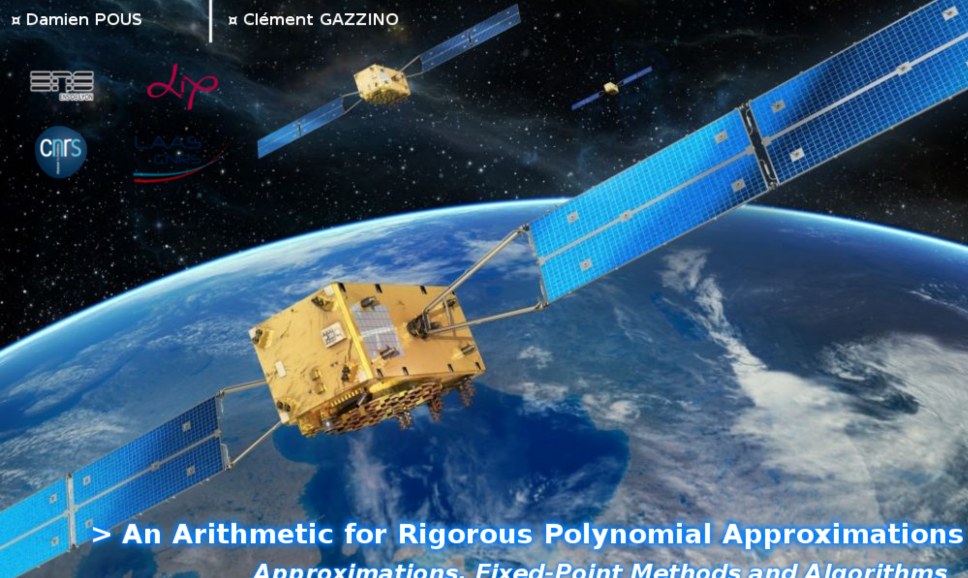
¤ Nicolas BRISEBARRE
¤ Mioara JOLDES
¤ Damien POUS
¤ Paulo Ricardo ARANTES GILZ
¤ Florent BREHARD
¤ Clément GAZZINO

| Lyon, RAIM 2017 _



# > An Arithmetic for Rigorous Polynomial Approximations

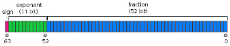
## *Approximations, Fixed-Point Methods and Algorithms for Function Space Problems*

*Credit: Galileo / ESA*

floating-point scientific computing

hardware implementation

robust IEEE 754 format




correctly rounded arithmetic operations

hugely optimized Floating-Point Units (FPU) in current processors

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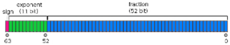


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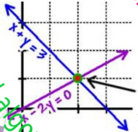
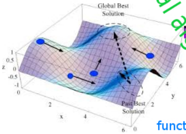
numerical algorithms

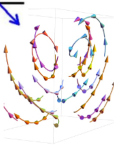
linear algebra



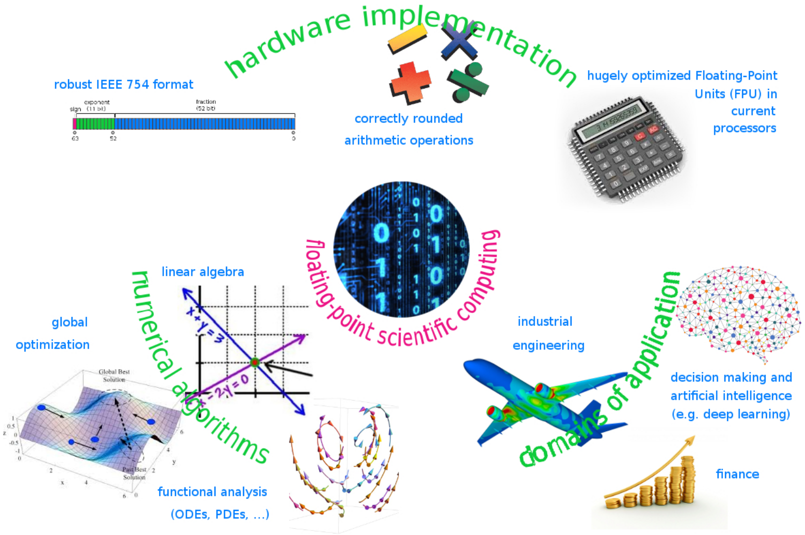

global optimization




functional analysis (ODEs, PDEs, ...)



# Power of Floating-Point Scientific Computing...



# ...but also Limits and Threats!

numerical
errors

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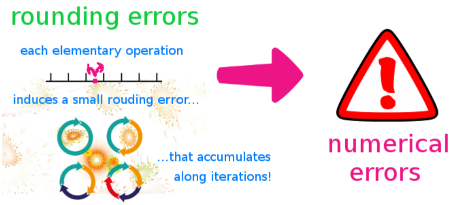

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each elementary operation

induces a small rouding error...

...that accumulates along iterations!

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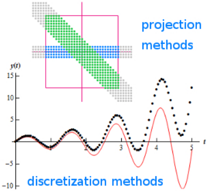
## approximation errors



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division by zero!

unsafe type casting!

binary64 » binary32

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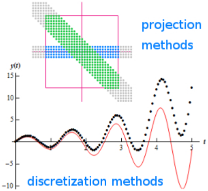

projection methods

discretization methods

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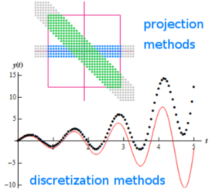


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Patriot missile failure in 1991 (28 killed)

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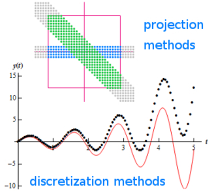


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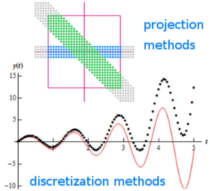


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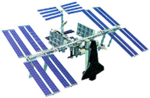

safety-critical engineering

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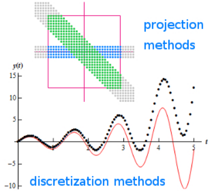


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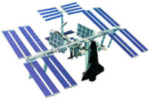
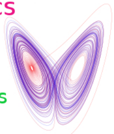
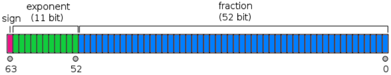


safety-critical engineering

computer-assisted mathematics

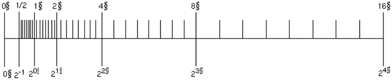


- Floating-Point Arithmetic:



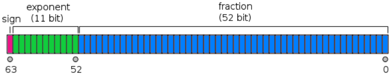

$$x = (-1)^{s} \cdot 1.\underbrace{1010011100\ldots1010}_{m} \cdot 2^{e}$$

Discretization of the real line:



# From Floating-Point Arithmetics to Interval Arithmetics

- Floating-Point Arithmetic:

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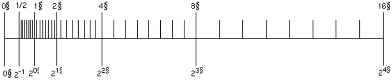


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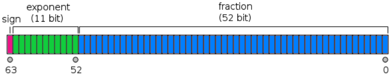


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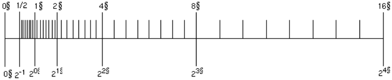


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- Interval Arithmetic

Overapprox reals by intervals

$\pi \in [3.14, 3.15]$ $\qquad$ $e \in [2.71, 2.72]$

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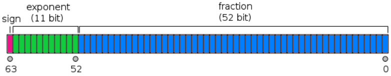


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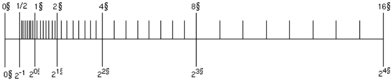


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Interval extension of arithmetic operators:

$$\pi - e \in [3.14 - 2.72, 3.15 - 2.71]$$

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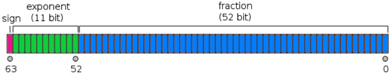


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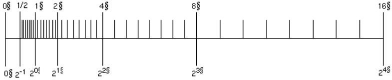

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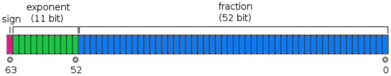


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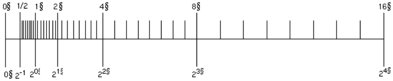

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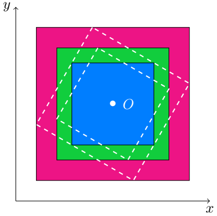
- Well-known limitations:

Wrapping effect



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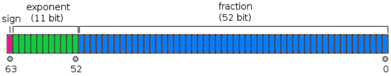


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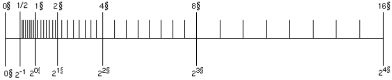


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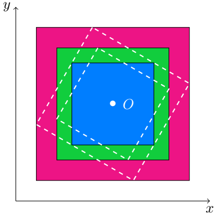
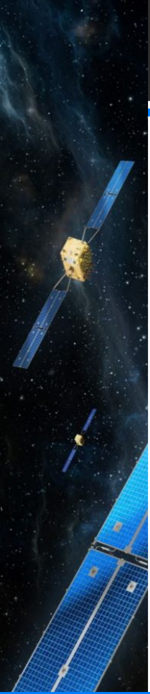



Loss of correlation

- $[-x, x] - [-x, x]$
  $= [-2x, 2x] \neq [0, 0]$.
- $\cos([0, 2\pi] + \varepsilon) - \cos([0, 2\pi])$
  $= [-1, 1] - [-1, 1] = [-2, 2]$
  but $|\cos(x + \varepsilon) - \cos(x)| \leq \varepsilon$.



# Outline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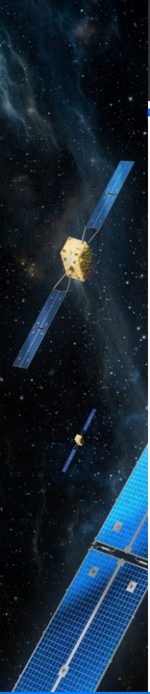
2 Rigorous Polynomial Approximations

3 A Posteriori Validation with Fixed-Points

4 Validated Solutions of Linear Differential Equations

5 Conclusion and Future Work

6 Some Extras



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**Theorem (Stone Weierstrass)**

*The family of polynomials is dense in the set of continuous functions over a compact interval.*

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- monomial basis
- fast computations
- related to initial conditions

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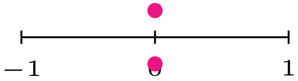

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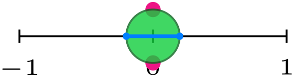

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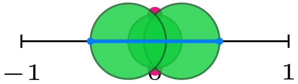

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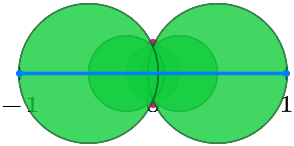
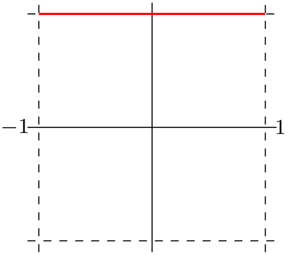


**Chebyshev Family of Polynomials**

$$T_0(X) = 1,$$
$$T_1(X) = X,$$
$$T_{n+2}(X) = 2XT_{n+1}(X) - T_n(X).$$

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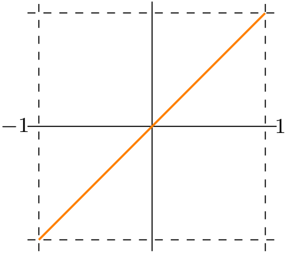


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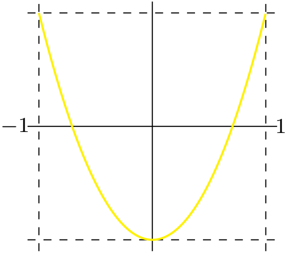


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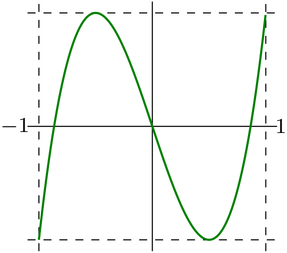


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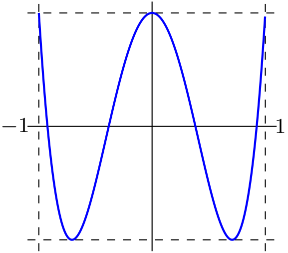


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$$T_4(X) = 8X^4 - 8X^2 + 1$$

# Chebyshev Polynomials and Series

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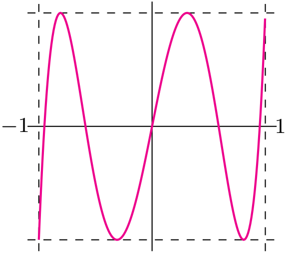


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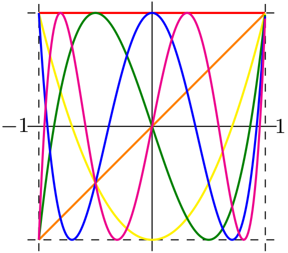


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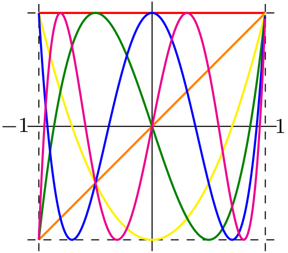


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### **Convergence Theorems**

- If $f \in \mathcal{C}^k$, $\widehat{f}^{[N]} \to f$ in $O(N^{-k})$.
- If $f$ analytic, $\widehat{f}^{[N]} \to f$ exponentially fast.

**Definition**

A pair $(P, \varepsilon) \in \mathbb{R}[X] \times \mathbb{R}_+$ is a rigorous polynomial approximation (RPA) of $f$ for a given norm $\|\cdot\|$ if $\|f - P\| \leq \varepsilon$.

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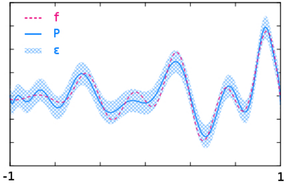


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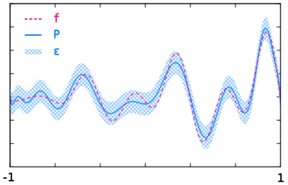


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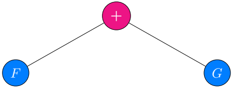
**Example:**

$$r(t) = f(t) + g(t)$$

**Some elementary operations:**

- $(P, \varepsilon) + (Q, \eta) := (P + Q, \varepsilon + \eta)$,



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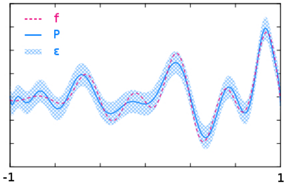


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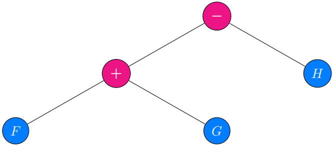
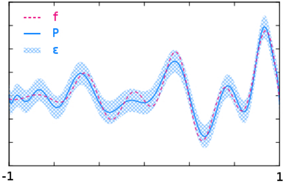


# Rigorous Polynomial Approximations

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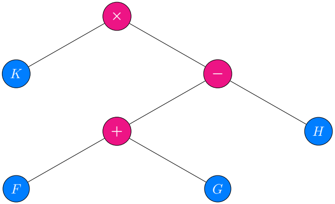

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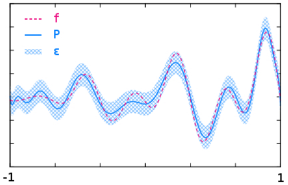


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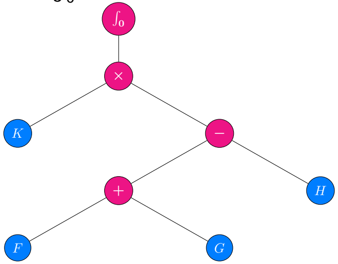


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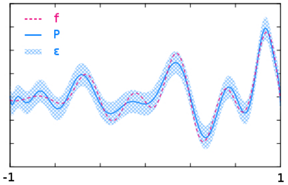


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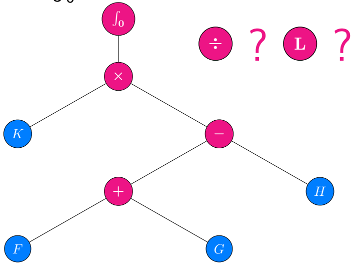
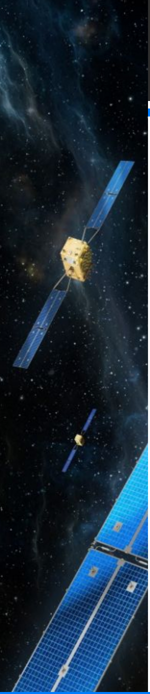


# Outline



1 Introduction

2 Rigorous Polynomial Approximations

3 A Posteriori Validation with Fixed-Points

4 Validated Solutions of Linear Differential Equations

5 Conclusion and Future Work

6 Some Extras

**Main Idea: A Posteriori Validation**

Reformulate the problem as a fixed-point equation $\mathbf{T} \cdot x = x$ over metric space $(X, d)$ and obtain $x$ candidate approximation of exact solution $x^*$.

- Find **rigorous** error bound $\|x - x^*\|$.

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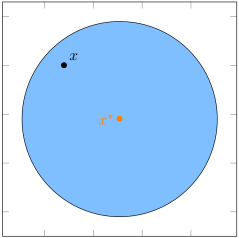
If $(X, d)$ is complete and $\mathbf{T}$ **contracting** of ratio $\mu < 1$,

- Then $\mathbf{T}$ admits a unique fixed-point $x^*$,



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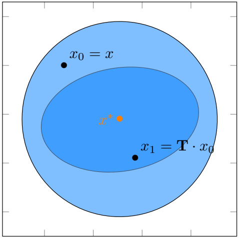

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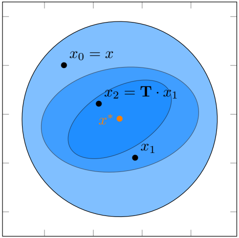

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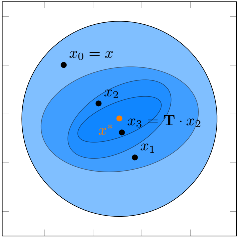

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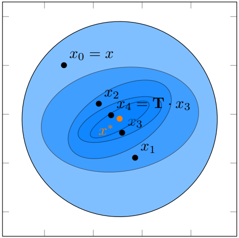

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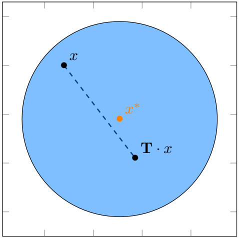
$$\frac{d(x, \mathbf{T} \cdot x)}{1 + \mu} \leq d(x, x^*) \leq \frac{d(x, \mathbf{T} \cdot x)}{1 - \mu}.$$



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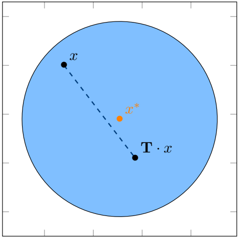
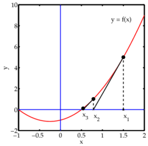



### Quasi-Newton Method for $\mathbf{F} \cdot x = 0$

Obtain $\mathbf{A} \approx (\mathrm{D}\mathbf{F})^{-1}$ in order to define:

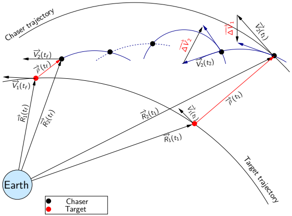
$$\mathbf{T} \cdot x = x - \mathbf{A} \cdot \mathbf{F} \cdot x.$$

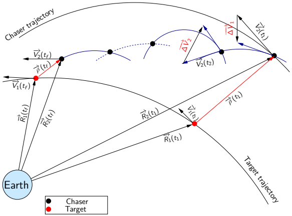
# An Example: Tschauner and Hempel Equation

Relative Motion in Keplerian Dynamics

**Reduced Equation**

$$z'' + \left(4 - \frac{3}{1 + e\cos\nu}\right) z = c$$



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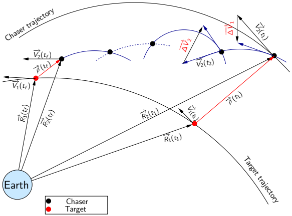


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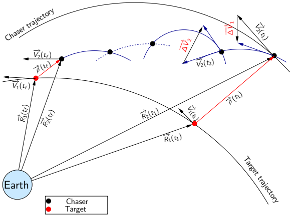


### To Do List

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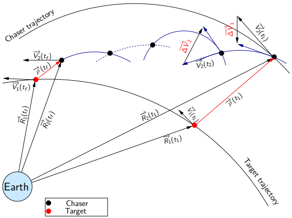


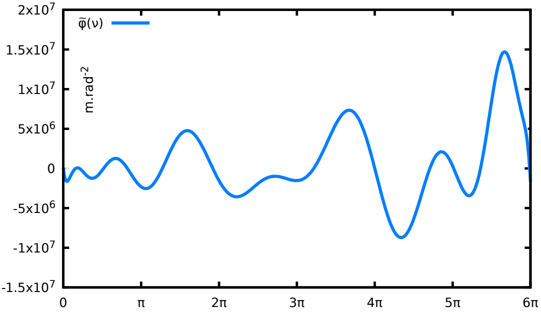

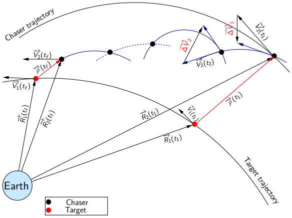


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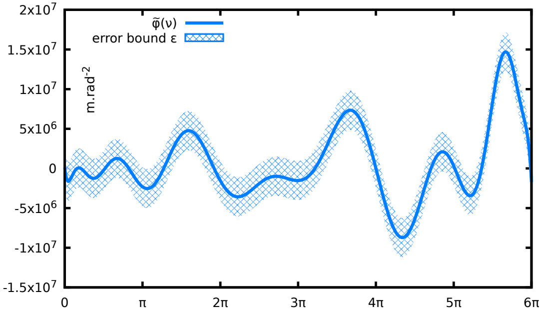

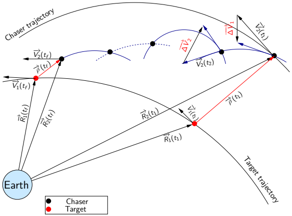


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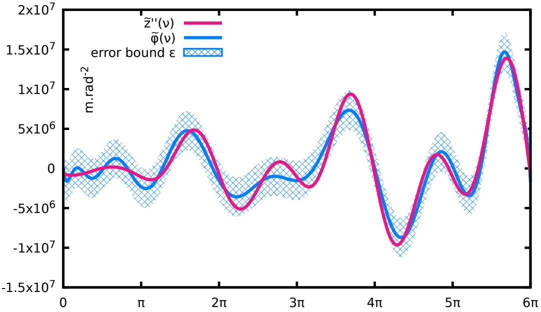

**Approximation of $x \mapsto 4 - \frac{3}{1+e\cos x}$**

✓ RPA for $x \mapsto \cos x$:

$$0.77\,T_0(x) - 0.23\,T_2(x) + 0.005\,T_4(x) \pm 4.2\cdot 10^{-5}$$

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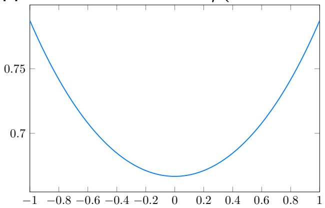
✓ RPA for $x \mapsto 1 + 0.5\cos x$:

$$1.38\,T_0(x) - 0.11\,T_2(x) + 0.002\,T_4(x) \pm 2.1 \cdot 10^{-5}$$

**Approximation of $x \mapsto 4 - \frac{3}{1+e\cos x}$**

✓ Approximation of $x \mapsto 1/(1+0.5\cos x)$:



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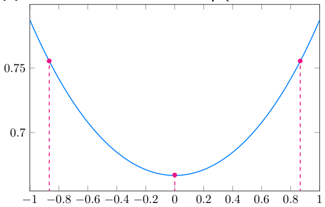

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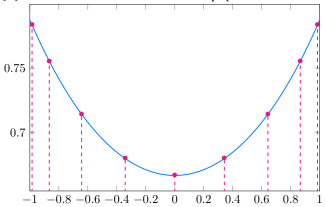

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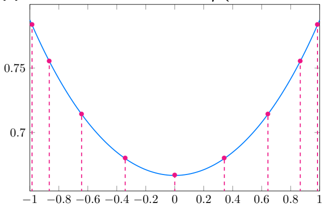



$\varphi = 0.73\,T_0(x) + 0.06\,T_2(x) \quad \approx 1/(1+0.5\cos x)$

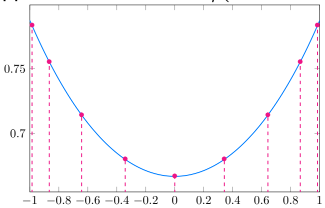
## Division: $g/f$ $(f \neq 0)$

- Solve $\mathbf{F} \cdot \varphi = f\varphi - g = 0$

$$(\mathrm{D}\mathbf{F})_\varphi \cdot h = f h \qquad (\mathrm{D}\mathbf{F})_\varphi^{-1} \cdot h = f^{-1} h$$

## Approximation of $x \mapsto 4 - \frac{3}{1+e\cos x}$

✓ Approximation of $x \mapsto 1/(1 + 0.5\cos x)$:

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$$\mathbf{T} \cdot \varphi = \varphi \qquad \mathbf{T} \cdot \varphi = \varphi - f_0(f\varphi - g)$$

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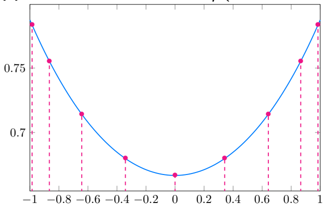


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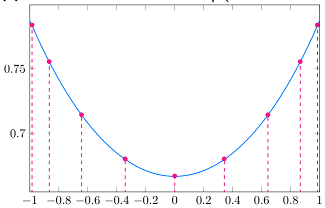


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# Fixed-Point Based Validation

## Application to Division

### Division: $g/f \;\; (f \neq 0)$

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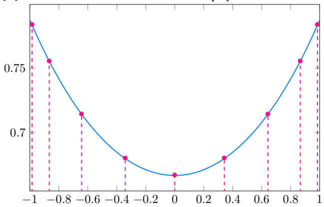


$\varphi = 0.73\,T_0(x) + 0.06\,T_2(x) \qquad \approx 1/(1+0.5\cos x)$

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## Division: $g/f$ ($f \neq 0$)

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$$\frac{\|f_0(f\varphi - g)\|}{1 + \mu} \leq \left\|\varphi - \frac{g}{f}\right\| \leq \frac{\|f_0(f\varphi - g)\|}{1 - \mu}$$

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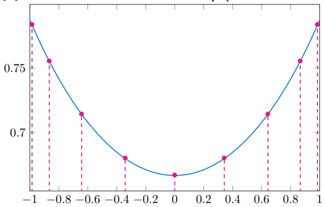


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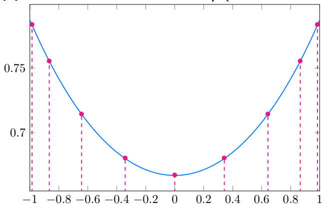


$\varphi = 0.73\,T_0(x) + 0.06\,T_2(x) \quad \approx 1/(1 + 0.5\cos x)$

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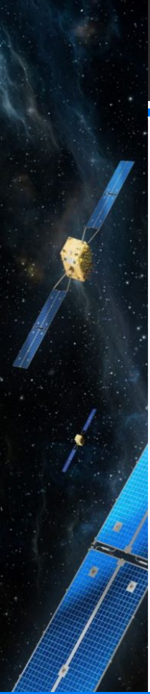
✓ RPA for $x \mapsto 1/(1 + 0.5\cos x)$:

$$0.73\,T_0(x) + 0.06\,T_2(x) \pm 1.3 \cdot 10^{-3}$$

✓ RPA for $x \mapsto 4 - 3/(1 + 0.5\cos x)$:

$$1.82\,T_0(x) - 0.18\,T_2(x) \pm 3.8 \cdot 10^{-3}$$



# Outline


1 Introduction

2 Rigorous Polynomial Approximations

3 A Posteriori Validation with Fixed-Points

4 Validated Solutions of Linear Differential Equations

5 Conclusion and Future Work

6 Some Extras

**LODE and Initial Value Problem**

$$
\begin{aligned}
&y^{(r)}(t) + \alpha_{r-1}(t) y^{(r-1)}(t) + \cdots + \alpha_1(t) y'(t) + \alpha_0(t) y(t) = g(t) \\
&y(-1) = v_0 \qquad y'(-1) = v_1 \qquad \ldots \qquad y^{(r-1)}(-1) = v_{r-1}
\end{aligned}
\tag{D}
$$

$t \in [-1, 1]$ $\alpha_i, g$ sufficiently regular ($\mathcal{C}^0$, RPA, **polynomial**)

### LODE and Initial Value Problem

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$t \in [-1, 1]$ $\alpha_i, g$ sufficiently regular ($\mathcal{C}^0$, RPA, **polynomial**)

### Integral Reformulation

Let $\varphi = y^{(r)}$, (D) becomes:

$$\varphi + \mathbf{K} \cdot \varphi = \psi, \qquad \text{(I)}$$

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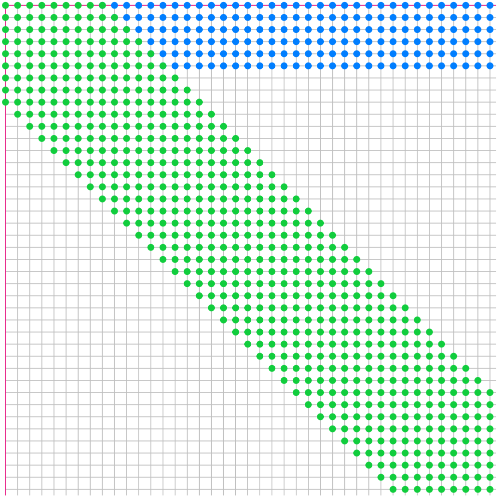
## Theorem (Picard-Lindelöf)

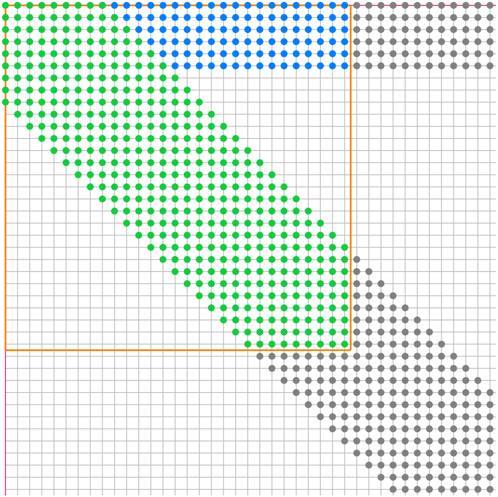
(I) (and hence (D)) has a **unique** solution.

# The Almost-Banded Structure of the Operator $\mathbf{K}$

## Matrix Representation in Chebyshev Basis



The infinite-dimensional operator $\mathbf{K}$.



The finite-dimensional truncation $\mathbf{K}^{[N]}$.

$$\mathbf{K} \cdot \varphi = t\left(4 - \frac{3}{1 + e\cos t}\right)\int_{t_0}^{t} \varphi(s)\mathrm{d}s + \left(-4 + \frac{3}{1 + e\cos t}\right)\int_{t_0}^{t} s\varphi(s)\mathrm{d}s$$

$$\mathbf{K} \cdot \varphi \approx t(1.82 - 0.18 T_2(t)) \int_{t_0}^{t} \varphi(s) \mathrm{d}s + (-1.82 + 0.18 T_2(t)) \int_{t_0}^{t} s\varphi(s) \mathrm{d}s$$

$$\mathbf{K}\cdot\varphi \approx \underbrace{(1.73\,T_1(t) - 0.09\,T_3(t))}_{\beta_0(t)} \int_{t_0}^{t} \varphi(s)\mathrm{d}s + \underbrace{(-1.82 + 0.18\,T_2(t))}_{\beta_1(t)} \int_{t_0}^{t} s\varphi(s)\mathrm{d}s$$

# The Almost-Banded Structure of the Operator $\mathbf{K}$

## Example with Tschauner-Hempel Equation

$$\mathbf{K}\cdot\varphi \approx \underbrace{(1.73\,T_1(t) - 0.09\,T_3(t))}_{\beta_0(t)} \int_{t_0}^{t} \varphi(s)\mathrm{d}s + \underbrace{(-1.82 + 0.18\,T_2(t))}_{\beta_1(t)} \int_{t_0}^{t} s\varphi(s)\mathrm{d}s$$

| | | | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1.342500 | -0.606667 | -0.326250 | 0.364000 | -0.155417 | 0.086667 | -0.056875 | 0.040444 | -0.030333 | 0.023636 | -0.018958 | 0.015556 | -0.013000 | 0.011030 | -0.009479 | 0.008235 | -0.007222 | 0.006386 | -0.005687 | 0.005098 | -0.004596 |
| 1.730000 | -0.652500 | -0.576667 | 0.438125 | -0.115333 | 0.070208 | -0.049429 | 0.036042 | -0.027460 | 0.021625 | -0.017475 | 0.014417 | -0.012098 | 0.010298 | -0.008872 | 0.007723 | -0.006784 | 0.006007 | -0.005356 | 0.004806 | -0.004336 |
| 0.320000 | 0.060000 | -0.271458 | -0.036000 | 0.093833 | -0.008571 | 0.004500 | -0.004000 | 0.003000 | -0.002338 | 0.001875 | -0.001538 | 0.001286 | -0.001091 | 0.000937 | -0.000814 | 0.000714 | -0.000632 | 0.000562 | -0.000504 | 0.000455 |
| -0.090000 | 0.109583 | 0.030000 | -0.137375 | 0.006000 | 0.036042 | 0.002571 | -0.002625 | 0.001429 | -0.001125 | 0.000909 | -0.000750 | 0.000629 | -0.000536 | 0.000462 | -0.000402 | 0.000353 | -0.000313 | 0.000279 | -0.000250 | 0.000226 |
| -0.022500 | 0 | 0.052917 | 0 | -0.065167 | 0 | 0.024036 | 0 | -0.000536 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | -0.003750 | 0 | 0.028375 | 0 | -0.040327 | 0 | 0.016104 | 0 | -0.000402 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | -0.001875 | 0 | 0.018167 | 0 | -0.027527 | 0 | 0.011548 | 0 | -0.000312 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 0 | -0.001125 | 0 | 0.012708 | 0 | -0.020021 | 0 | 0.008688 | 0 | -0.000250 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 | -0.000750 | 0 | 0.009411 | 0 | -0.015230 | 0 | 0.006774 | 0 | -0.000205 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 | 0 | -0.000536 | 0 | 0.007257 | 0 | -0.011981 | 0 | 0.005431 | 0 | -0.000170 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 | 0 | 0 | -0.000402 | 0 | 0.005770 | 0 | -0.009675 | 0 | 0.004451 | 0 | -0.000144 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | -0.000312 | 0 | 0.004699 | 0 | -0.007978 | 0 | 0.003715 | 0 | -0.000124 | 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | -0.000250 | 0 | 0.003902 | 0 | -0.006692 | 0 | 0.003147 | 0 | -0.000107 | 0 | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | -0.000205 | 0 | 0.003292 | 0 | -0.005694 | 0 | 0.002701 | 0 | -0.000094 | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | -0.000170 | 0 | 0.002815 | 0 | -0.004905 | 0 | 0.002343 | 0 | -0.000083 | 0 | 0 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | -0.000144 | 0 | 0.002435 | 0 | -0.004269 | 0 | 0.002052 | 0 | -0.000074 | 0 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | -0.000124 | 0 | 0.002127 | 0 | -0.003749 | 0 | 0.001812 | 0 | -0.000066 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | -0.000107 | 0 | 0.001874 | 0 | -0.003319 | 0 | 0.001612 | 0 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | -0.000094 | 0 | 0.001663 | 0 | -0.002959 | 0 | 0.001443 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | -0.000083 | 0 | 0.001487 | 0 | -0.002655 | 0 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | -0.000074 | 0 | 0.001337 | 0 | -0.002395 |

# Approximate Solution to Tschauner-Hemple Equation

- We want to solve $z''(t) + \left(4 - \frac{3}{1+0.5\cos t}\right) z(t) = c$ with $z(-1) = 0$, $z'(-1) = 1$ and $c = 1$.

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- $\psi \approx -0.82\,T_0 - 1.73\,T_1 + 0.18\,T_2 + 0.09\,T_3$.

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$$\begin{aligned}\varphi = {} & -0.6\,T_0 - 1.19\,T_1 + 0.62\,T_2 + 0.17\,T_3 - 0.05\,T_4 - 0.01\,T_5 \\ & + 2.1 \cdot 10^{-3}\,T_6 + 3.2 \cdot 10^{-3}\,T_7 - 5.8 \cdot 10^{-5}\,T_8 - 7.6 \cdot 10^{-6}\,T_9 + 1.2 \cdot 10^{-6}\,T_{10} \\ & + 1.4 \cdot 10^{-7}\,T_{11} - 1.9 \cdot 10^{-8}\,T_{12} - 2.0 \cdot 10^{-9}\,T_{13} + 2.6 \cdot 10^{-10}\,T_{14} + 2.5 \cdot 10^{-11}\,T_{15} \\ & - 3.0 \cdot 10^{-12}\,T_{16} - 2.6 \cdot 10^{-13}\,T_{17} + 3.0 \cdot 10^{-14}\,T_{18} + 2.5 \cdot 10^{-15}\,T_{19} - 2.6 \cdot 10^{-16}\,T_{20}\end{aligned}$$

# Designing the Newton-like Operator $\mathbf{T}$

**Construct $\mathbf{T}$: To-Do List**

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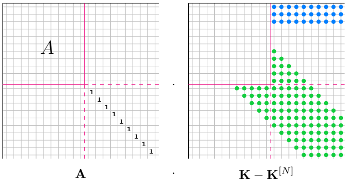
**Truncation error:**

- Determines the minimal value of $N$ we can choose.



$$\mathbf{L} \cdot y = y^{(r)}(t) + \alpha_{r-1}(t) y^{(r-1)}(t) + \cdots + \alpha_1(t) y'(t) + \alpha_0(t) y(t) = g(t)$$
$$y(-1) = v_0 \qquad y'(-1) = v_1 \qquad \ldots \qquad y^{(r-1)}(-1) = v_{r-1} \qquad \text{(D)}$$

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1. Integral reformulation: $\varphi + \mathbf{K} \cdot \varphi = \psi$ with $\varphi = y^{(r)}$,
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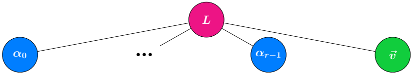

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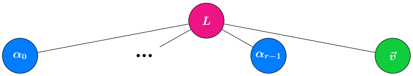
- Extension to *Boundary Value Problems* (BVP)



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$$\frac{6.48 \cdot 10^{-16}}{1 + \mu} \leq \|\varphi - \varphi^*\| \leq \frac{6.48 \cdot 10^{-16}}{1 - \mu}$$

# Solution to Tschauner and Hempel Equations

## Bring our Example to the End

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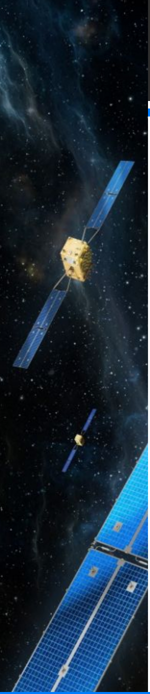
⚠ Take into account approximation error of coefficient!



# Outline

1. Introduction
2. Rigorous Polynomial Approximations
3. A Posteriori Validation with Fixed-Points
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5. **Conclusion and Future Work**
6. Some Extras

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¤ Nicolas BRISEBARRE
¤ Mioara JOLDES
¤ Damien POUS
¤ Paulo Ricardo ARANTES GILZ
¤ Florent BREHARD
¤ Clément GAZZINO

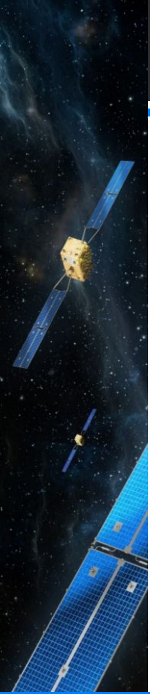

| Lyon, RAIM 2017 _

# Thank you!

## > An Arithmetic for Rigorous Polynomial Approximations

*Approximations, Fixed-Point Methods and Algorithms for Function Space Problems*

*Credit: Galileo / ESA*



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dessin inclusion boule image

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- Check $\Delta \geq 0 \quad \to \quad 0 \leq r_{min} \leq r_{max}$.
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# Fixed-Point Based Validation

## Application to Division and Square Root

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- $\mathbf{T}$ **non-linear**, $\|(\mathrm{D}\mathbf{T})_\varphi\| = \|1 - f_0\varphi\|$.

$\exists r > 0 \cdot \quad \mathbf{T} : \overline{B}(\varphi, r) \to \overline{B}(\varphi, r)$?

$$\|\mathbf{T} \cdot \varphi - \varphi\| + r \sup_{\psi \in \overline{B}(\varphi, r)} \|(\mathrm{D}\mathbf{T})_\psi\| < r$$

- Check $\Delta \geq 0$ $\quad \rightarrow \quad$ $0 \leq r_{\min} \leq r_{\max}$.
- Check $\mu = \|1 - f_0\varphi\| + \|f_0\| r_{\min} < 1$.

$$\frac{\|f_0(\varphi^2 - f)/2\|}{1+\mu} \leq \|\varphi - \sqrt{f}\| \leq \frac{\|f_0(\varphi^2 - f)/2\|}{1-\mu}$$

**Truncation Error**

$$\|\mathbf{A} \cdot (\mathbf{K} - \mathbf{K}^{[N]})\| = \sup_{i \geq 0} \|\mathbf{A} \cdot (\mathbf{K} - \mathbf{K}^{[N]}) \cdot T_i\|$$

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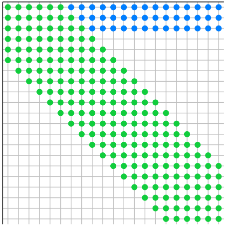

$\mathbf{K}$

**Truncation Error**

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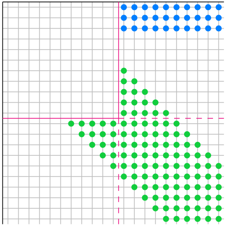
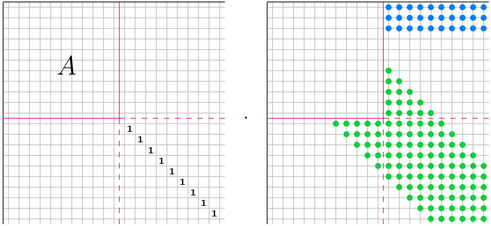


$\mathbf{K} - \mathbf{K}^{[N]}$

## Truncation Error

$$\|\mathbf{A}\cdot(\mathbf{K}-\mathbf{K}^{[N]})\| = \sup_{i\geq 0} \|\mathbf{A}\cdot(\mathbf{K}-\mathbf{K}^{[N]})\cdot T_i\|$$

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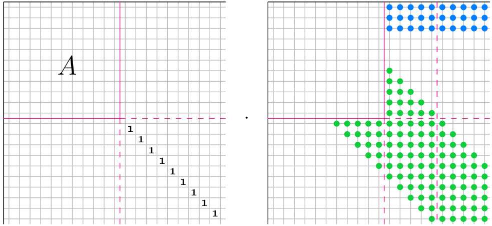


$\mathbf{A}$ $\cdot$ $\mathbf{K}-\mathbf{K}^{[N]}$

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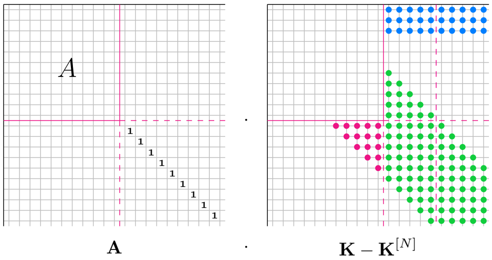


1. Direct computation.

**Truncation Error**

$$\|\mathbf{A}\cdot(\mathbf{K}-\mathbf{K}^{[N]})\| = \sup_{i\geq 0} \|\mathbf{A}\cdot(\mathbf{K}-\mathbf{K}^{[N]})\cdot T_i\|$$

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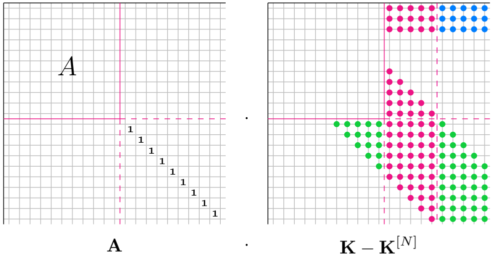
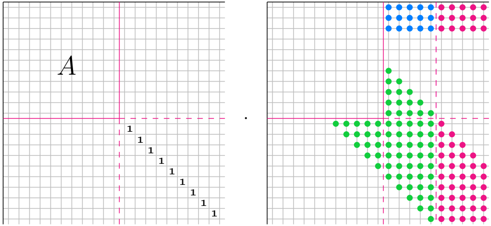
2. Direct computation.



**Truncation Error**

$$\|\mathbf{A} \cdot (\mathbf{K} - \mathbf{K}^{[N]})\| = \sup_{i \geq 0} \|\mathbf{A} \cdot (\mathbf{K} - \mathbf{K}^{[N]}) \cdot T_i\|$$

1. Direct computation.
2. Direct computation.
3. Bound the remaining *infinite* number of columns:




$\mathbf{A}$ · $\mathbf{K} - \mathbf{K}^{[N]}$

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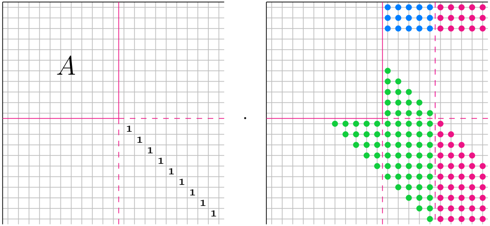


$\mathbf{A} \cdot \mathbf{K} - \mathbf{K}^{[N]}$

1. Direct computation.
2. Direct computation.
3. Bound the remaining *infinite* number of columns:
   - Using the bounds in $1/i$ and $1/i^2$: possibly large overestimations.

$$diag(i) \leq \frac{C}{i} \quad init(i) \quad \leq \frac{D}{i^2}$$

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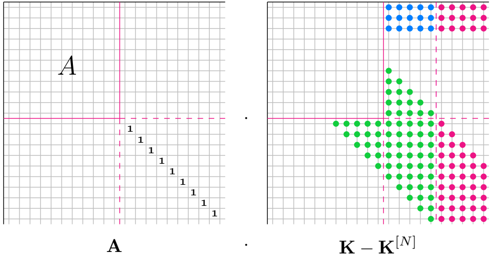
$$\|\mathbf{A}\cdot(\mathbf{K}-\mathbf{K}^{[N]})\| = \sup_{i\geq 0}\|\mathbf{A}\cdot(\mathbf{K}-\mathbf{K}^{[N]})\cdot T_i\|$$

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   - Using a first order difference method: differences in $1/i^2$ and $1/i^4$.

     $$diag(i) \leq diag(i_0) + \frac{C'}{i^2}$$

     $$init(i) \leq init(i_0) + \frac{D'}{i^4}$$

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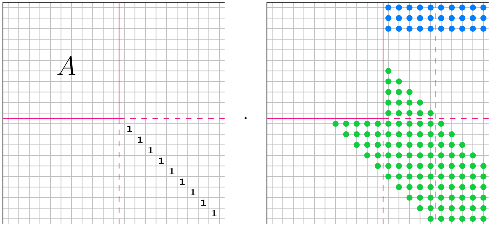


$\mathbf{A} \cdot \mathbf{K}-\mathbf{K}^{[N]}$

**Cost:** $O(N(h+d))$ **or** $O((h'+d')(h+d))$

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### Coupled LODEs and Initial Value Problem

$$Y^{(r)}(t) + A_{r-1}(t) \cdot Y^{(r-1)}(t) + \cdots + A_1(t) \cdot Y'(t) + A_0(t) \cdot Y(t) = G(t) \qquad \text{(p-D)}$$

$$A_k(t) = \begin{pmatrix} a_{k11}(t) & \cdots & a_{k1p}(t) \\ \vdots & \ddots & \vdots \\ a_{kp1}(t) & \cdots & a_{kpp}(t) \end{pmatrix} \qquad G(t) = \begin{pmatrix} g_1(t) \\ \vdots \\ g_p(t) \end{pmatrix}$$

$$t \in [-1,1] \qquad Y_i^{(k)}(-1) = v_{ik} \qquad i \in [\![1,p]\!], k \in [\![0,r-1]\!]$$

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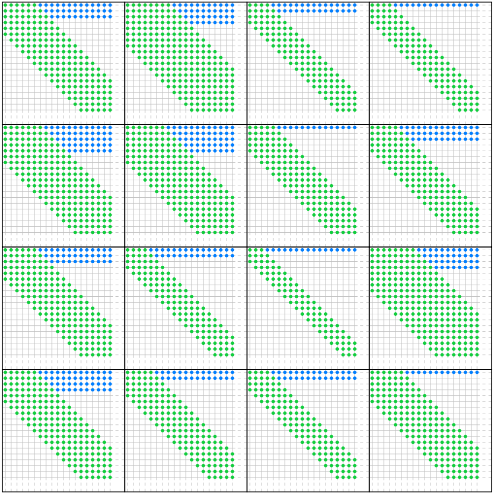
### Integral Reformulation

Posing $\Phi = Y^{(r)}$, System (p-D) is transformed into:

$$\Phi(t) + \int_{t_0}^{t} \begin{pmatrix} k_{11}(t,s) & \cdots & k_{1p}(t,s) \\ \vdots & \ddots & \vdots \\ k_{p1}(t) & \cdots & k_{pp}(t) \end{pmatrix} \cdot \Phi(s)\mathrm{d}s = \Psi(t) \qquad \text{(p-I)}$$

| | | | |
|---|---|---|---|
| $\mathbf{K}_{1,1}$ | $\mathbf{K}_{1,2}$ | $\mathbf{K}_{1,3}$ | $\mathbf{K}_{1,4}$ |
| $\mathbf{K}_{2,1}$ | $\mathbf{K}_{2,2}$ | $\mathbf{K}_{2,3}$ | $\mathbf{K}_{2,4}$ |
| $\mathbf{K}_{3,1}$ | $\mathbf{K}_{3,2}$ | $\mathbf{K}_{3,3}$ | $\mathbf{K}_{3,4}$ |
| $\mathbf{K}_{4,1}$ | $\mathbf{K}_{4,2}$ | $\mathbf{K}_{4,3}$ | $\mathbf{K}_{4,4}$ |

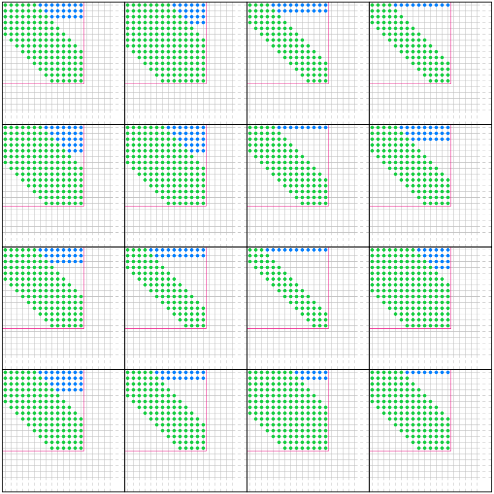
$\mathbf{K}$



$\mathbf{K}$

# The Almost-Banded Structure of the Operator $\mathbf{K}$

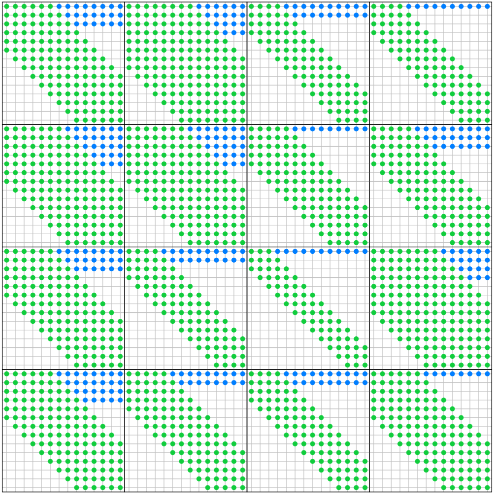
## Example in Dimension 4



$\mathbf{K}^{[N]}$

# The Almost-Banded Structure of the Operator $\mathbf{K}$

## Example in Dimension 4



$\mathbf{K}^{[N]}$

$\mathbf{K}^{[N]}$ (rearranged basis)

### Vector-Valued Metric and Contractions

$(X_1, d_1), \dots, (X_p, d_p)$ complete metric spaces.

- $d(x, y) = (d_1(x_1, y_1), \dots, d_p(x_p, y_p)) \in \mathbb{R}_+^p$ vector-valued metric.
- $f : X \to X$ is $\boldsymbol{\Lambda}$-Lipschitz for $\boldsymbol{\Lambda} \in \mathbb{R}_+^{p \times p}$ iff:

$$d(f(x), f(y)) \leq \boldsymbol{\Lambda} \cdot d(x, y) \qquad \forall x, y \in X$$

- $f : X \to X$ is a contraction if it is $\boldsymbol{\Lambda}$-Lipschitz for $\boldsymbol{\Lambda} \in \mathbb{R}_+^{p \times p}$ s.t. $\boldsymbol{\Lambda}^k \to 0$ as $k \to \infty$.

# Vector-Valued Fixed-Point Validation

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### Error Polytope

Let $\varepsilon = d(x, x^*)$ and $\eta = d(x, \mathbf{T} \cdot x)$:

$$\begin{aligned} (1 - \mathbf{\Lambda}) \cdot \varepsilon &\leq \eta \\ (1 + \mathbf{\Lambda}) \cdot \varepsilon &\geq \eta \\ \varepsilon &\geq 0 \end{aligned} \qquad \text{(P)}$$

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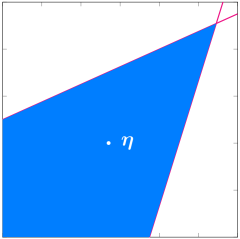

# Vector-Valued Fixed-Point Validation

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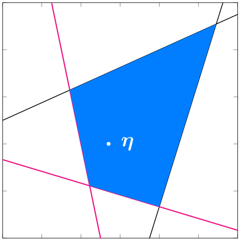

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