ELECTRONICS & DEFENSE

A DECIMAL MULTIPLE-PRECISION INTERVAL ARITHMETIC LIBRARY

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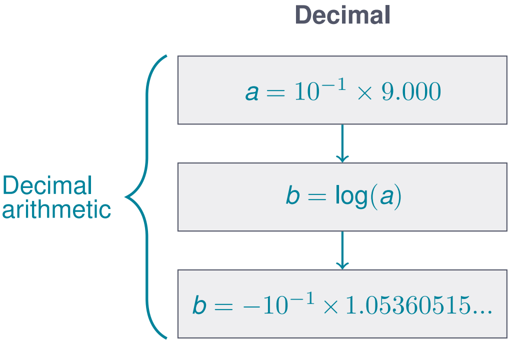
Arithmetic: Decimal vs Binary



Decimal

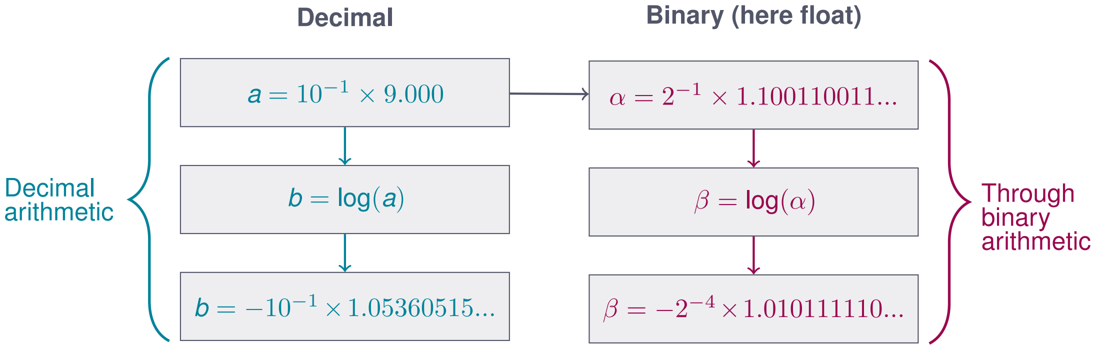
Decimal arithmetic

$$a = 10^{-1} \times 9.000$$

$$b = \log(a)$$

$$b = -10^{-1} \times 1.05360515\ldots$$

Arithmetic: Decimal vs Binary



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Decimal arithmetic

Binary (here float)

$\alpha = 2^{-1} \times 1.100110011...$

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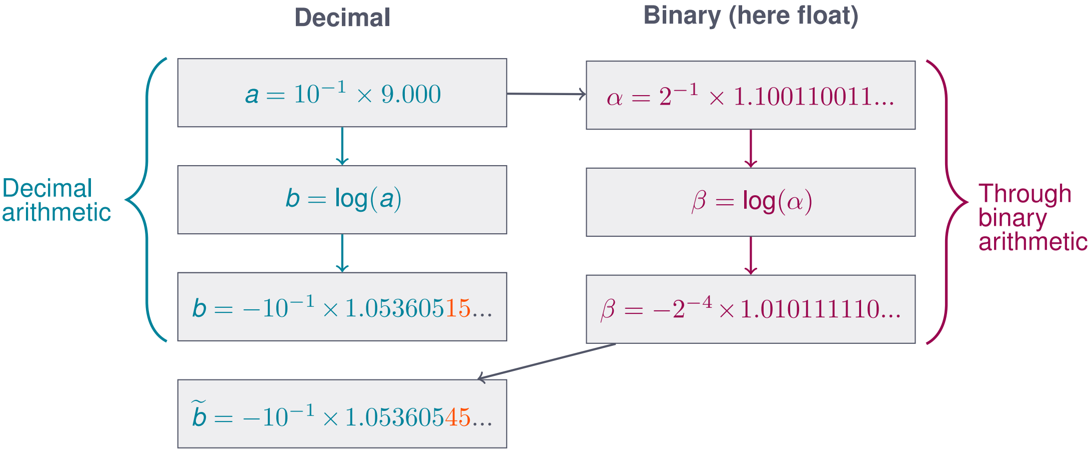
Through binary arithmetic

Arithmetic: Decimal vs Binary



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Motivations for our work

In what context a decimal multiple-precision interval library may be used?

- Financial applications
- Validation and test of systems subject to strong certification processes:
  - aerospace industry,
  - autonomous car...

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Why use multiple-precision?

- multiple-precision arithmetic: in opposition to fixed precision arithmetic, enables the user to choose the precision of each variable.
- The precision is only limited by the space and the time needed for the computation.
  - Increasing the precision may increase the accuracy.

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Why use interval arithmetic?

- interval arithmetic: represent a number with an interval, representable by machine numbers, containing its value. It gives us guarantees on the numerical result.

Our goal

Provide a decimal multiple-precision interval arithmetic through two libraries

- MPD: Multiple Precision Decimal
- MPDI: Multiple Precision Decimal Interval

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State of the art libraries

- multiple precision: BigInteger, BigDecimal
- interval arithmetic: JInterval

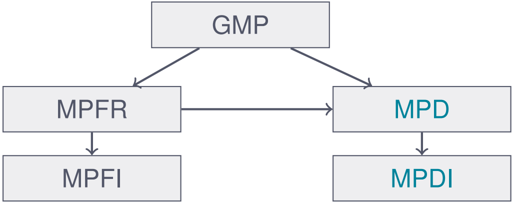
Our contributions

- Correctly rounded
- Reliable
- Fast

Our goal

Provide a decimal multiple-precision interval arithmetic through two libraries

- MPD: Multiple Precision Decimal
  - Using MPFR and GMP mathematical functions
- MPDI: Multiple Precision Decimal Interval
  - based on MPD the same way MPFI is based on MPFR



- GMP: GNU Multiple Precision Arithmetic Library
- MPFR: Multiple Precision Floating-Point Reliable Library
- MPFI: Multiple Precision Floating-Point Interval Library

MPD and MPDI types

Note on the IEEE Standard for Floating-Point Arithmetic (IEEE-754 2008)

- MPD arithmetic is IEEE 754 compatible, but not compliant (e.g. there is no quantum for now)
- MPD numbers are not normalized: a number does not have a unique representation in MPD

MPD type

$$MPD : 10^{F} \times n$$

- significand *n* is a GMP signed integer,
- exponent *F* is an integer,
- additional information structure (NaN, $\pm\infty$, ± 0 or number),
- the decimal precision *k* in digits

MPDI type

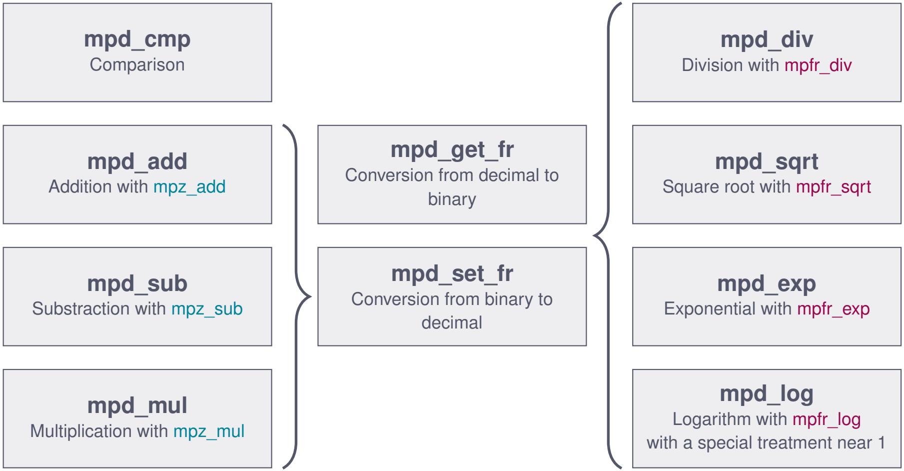
$$MPDI : [left, right]$$

- Interval of two MPD numbers *left* and *right*

Outline

1. Introduction
2. Mechanisms behind MPD
3. Mechanisms behind MPDI
4. Benchmark

MPD Library Architecture



mpd_cmp
Comparison

mpd_add
Addition with mpz_add

mpd_sub
Substraction with mpz_sub

mpd_mul
Multiplication with mpz_mul

mpd_get_fr
Conversion from decimal to binary

mpd_set_fr
Conversion from binary to decimal

mpd_div
Division with mpfr_div

mpd_sqrt
Square root with mpfr_sqrt

mpd_exp
Exponential with mpfr_exp

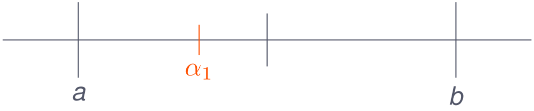
mpd_log
Logarithm with mpfr_log
with a special treatment near 1

- Two central functions
  - conversion binary → decimal
  - conversion decimal → binary
- Two groups of functions
  - with GMP
  - with MPFR

Conversion algorithm: binary to decimal

Indecision with the rounding mode to the nearest

- *a* and *b* two representable decimal numbers in the current precision



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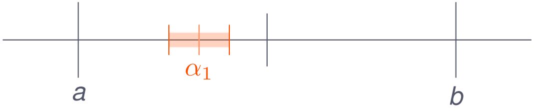

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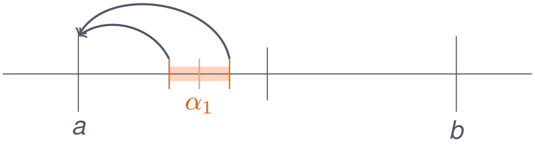

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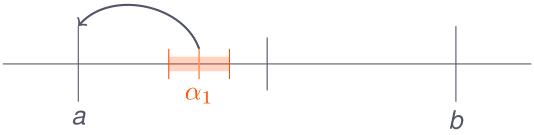
    - **Ziv's loop**



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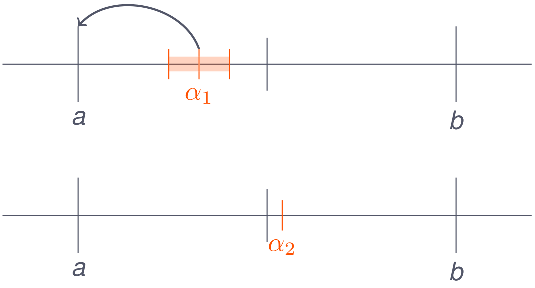

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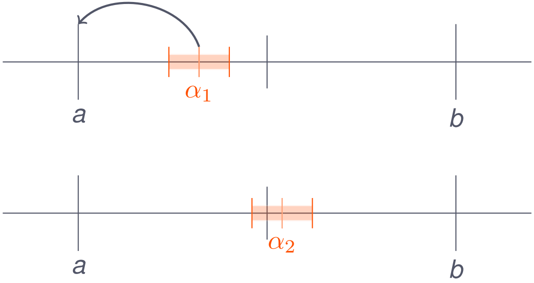

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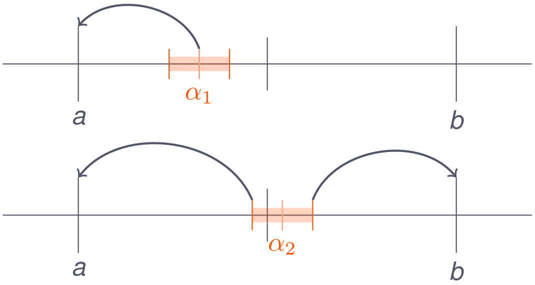

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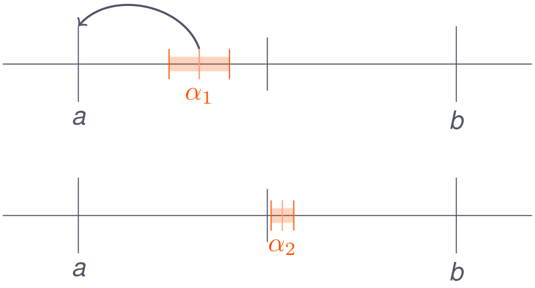

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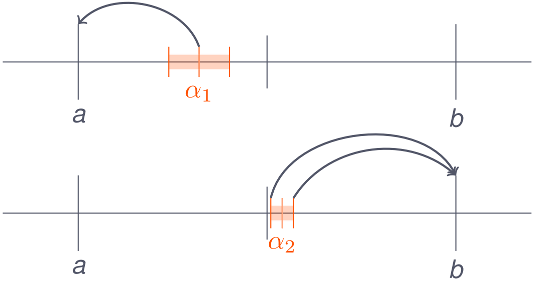

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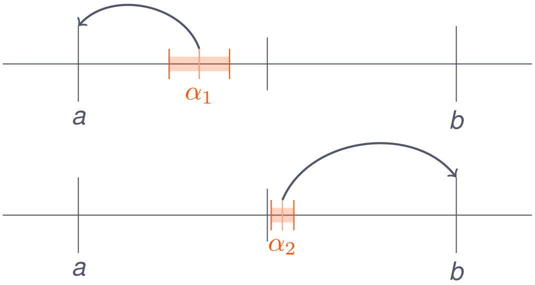

Multiplication algorithm

Simple algorithm without conversion

We want to compute $10^{F} \times n = (10^{F_1} \times n_1) \times (10^{F_2} \times n_2)$

$F = F_1 + F_2$	add the exponents
$t =$ **mpz_mul**(n_1, n_2)	multiplication of the significand with the GMP function
$\widetilde{n} =$ **mpd_set_z**(t)	Round the result to the decimal precision k

Addition algorithm

Simple algorithm without conversion

We want to compute $10^F \times n = (10^{F_1} \times n_1) + (10^{F_2} \times n_2)$

- if needed, align the two decimal numbers with multiplications (instead of divisions as in binary)
- set the exponent F
- $t =$ **mpz_add**(n_1, n_2) add the significands
- $\widetilde{n} =$ **mpd_set_z**(t) round the result to the decimal precision k

Square root algorithm

Decimal (MPD)

Binary (MPFR)

$a = 10^F \times n$
with k digit precision

Square root algorithm

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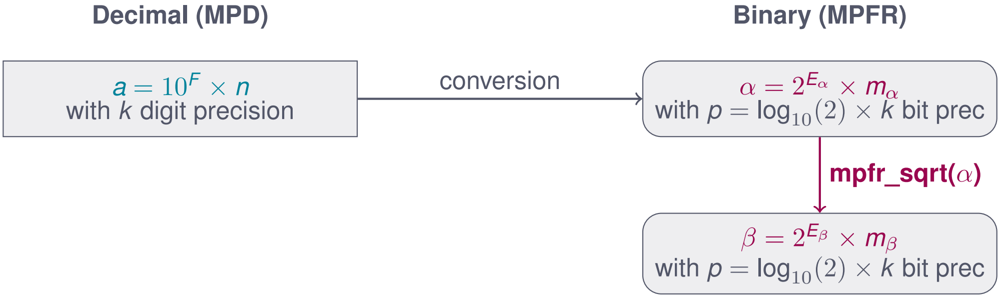
conversion

$\alpha = 2^{E_\alpha} \times m_\alpha$
with $p = \log_{10}(2) \times k$ bit prec

Square root algorithm



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mpfr_sqrt(α)

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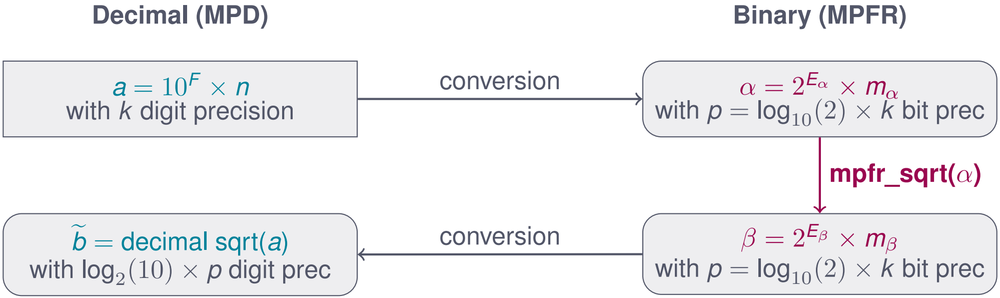

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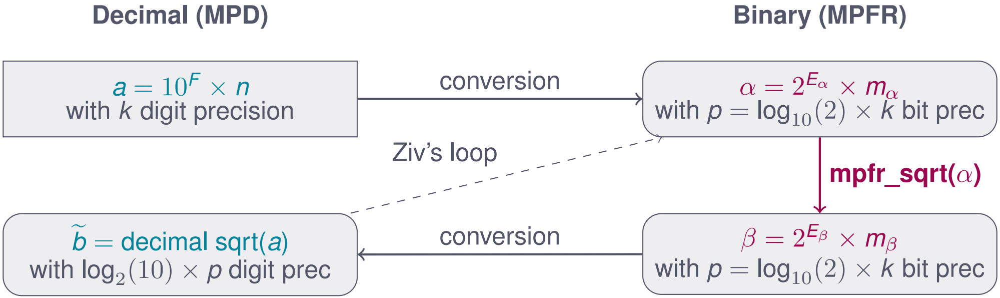
$\widetilde{b} = \text{decimal sqrt}(a)$
with $\log_2(10) \times p$ digit prec

Square root algorithm



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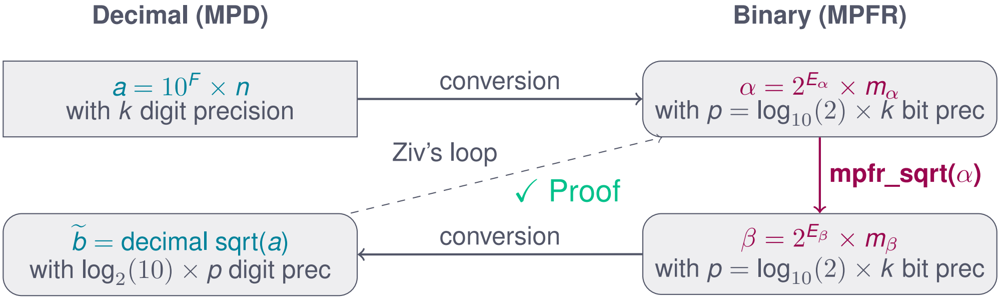

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Ziv's loop

✓ Proof

$\widetilde{b}$ = decimal sqrt(a)
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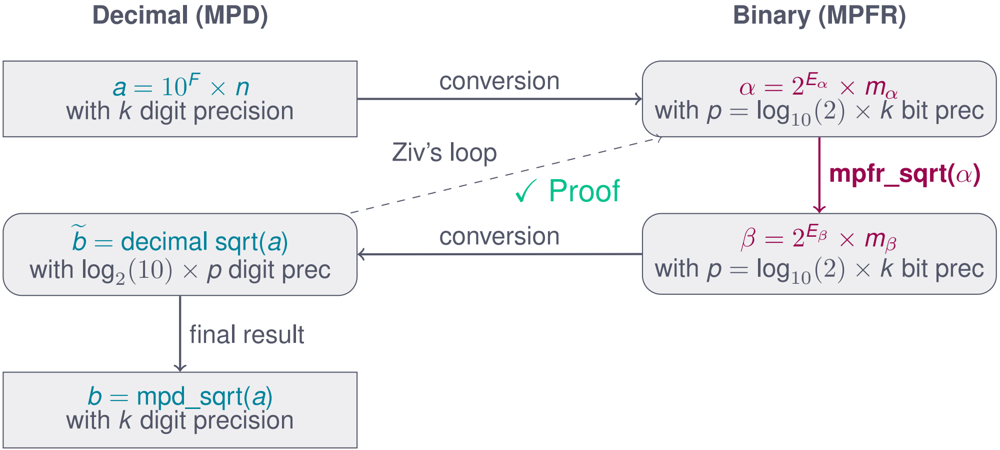

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✓ Proof

mpfr_sqrt(α)

$\widetilde{b} =$ decimal sqrt(a)
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conversion

$\beta = 2^{E_\beta} \times m_\beta$
with $p = \log_{10}(2) \times k$ bit prec

final result

$b =$ mpd_sqrt(a)
with k digit precision

Logarithm algorithm

$$\log(a) = \log(\alpha(1+\varepsilon)) = \log(\alpha) \cdot \left(1 + \frac{1}{\log(\alpha)} \cdot \frac{\log(1+\varepsilon)}{\varepsilon} \cdot \varepsilon\right)$$

Problem when α is around 1

- Afar from 1
  - implement the decimal logarithm with **mpfr_log** in the same way as the square root
- Around 1
  - compute in decimal $b = a - 1$ with **mpd_sub**
  - convert the decimal b into the binary β with **mpd_set_fr**
  - perform the logarithm operation with **mpfr_log1p**
    - this function implements $\text{log1p}(\beta) = \log(1+\beta)$

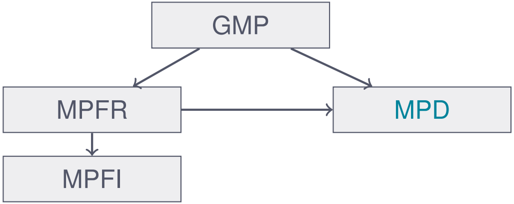
Outline

- 1 Introduction
- 2 Mechanisms behind MPD
- 3 **Mechanisms behind MPDI**
- 4 Benchmark

Multiple Precision Decimal Interval

So far we have:

- MPFR: a correctly rounded multiple precision binary library
- MPFI: a correctly rounded interval multiple precision binary library
- MPD: a correctly rounded multiple precision decimal library



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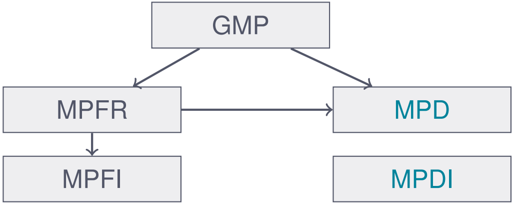


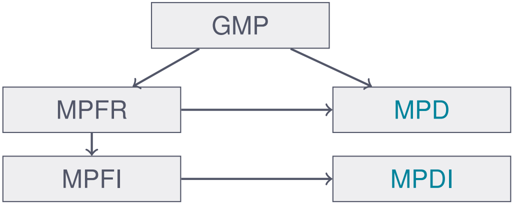
We want to implement:

- MPDI: a correctly rounded interval multiple precision decimal library

Multiple Precision Decimal Interval

Two solutions:

- Copy MPD methodology
  - convert the decimal interval $[a, b]$ into the binary one $[\alpha, \beta]$
  - compute the binary result with MPFI functions



GMP

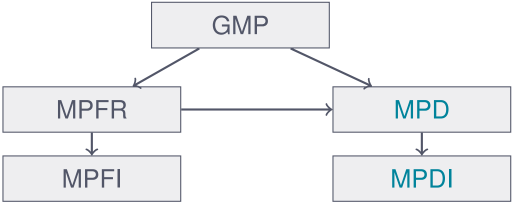
MPFR

MPD

MPFI

MPDI

- Take inspiration from the MPFI code
  - use MPD functions with directed rounding to compute the decimal interval



GMP

MPFR

MPD

MPFI

MPDI

List of implemented functions

MPD implementation

- init and clear functions
- set default and current precision
- mpd_set: convert in decimal from a decimal (MPD), a binary floating point (MPFR), a binary integer (GMP).
- mpd_get: convert a decimal into a binary
- arithmetic functions: add, sub, mul, div, sqrt.
- transcendental functions: log, exp.
- comparison functions: cmp, cmp_ui.
- absolute value and negation.

MPDI implementation

- init and clear
- set the prec, set the default prec
- mpdi_set: set a decimal interval MPDI from a decimal number MPD
- arithmetic functions: add, sub, mul, div, sqrt.
- transcendental functions: log, exp.

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3 Mechanisms behind MPDI

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Do we have the expected output?

Decimal MPD
prec $k = 8$

$a = 10^{-1} \times 9.000$

$\downarrow$

$b = \text{mpd\_log}(a)$

$\downarrow$

$b = -10^{-1} \times 1.0536052$

Binary MPFR
prec $p = 24$

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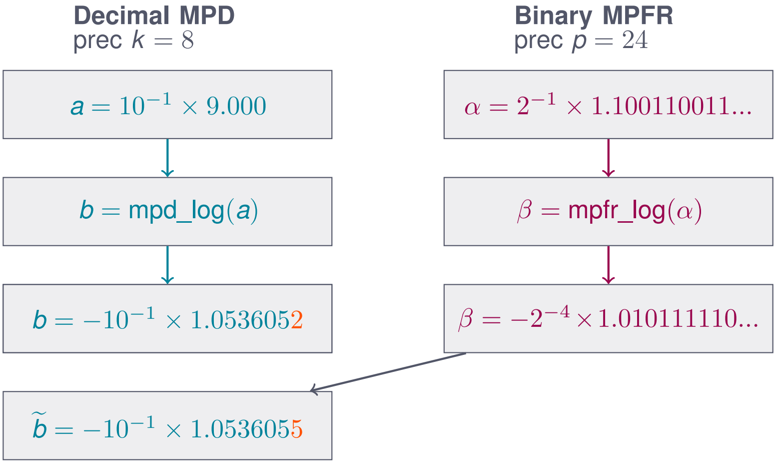

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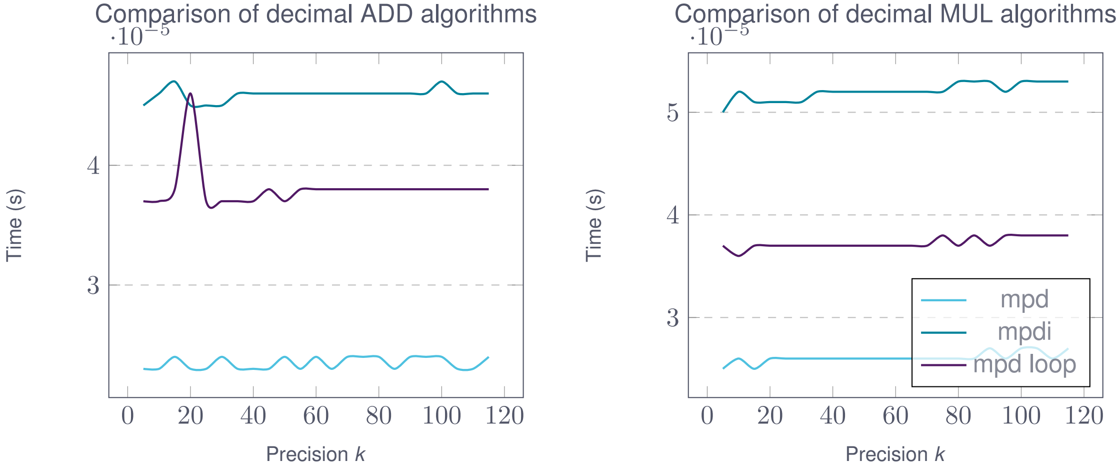
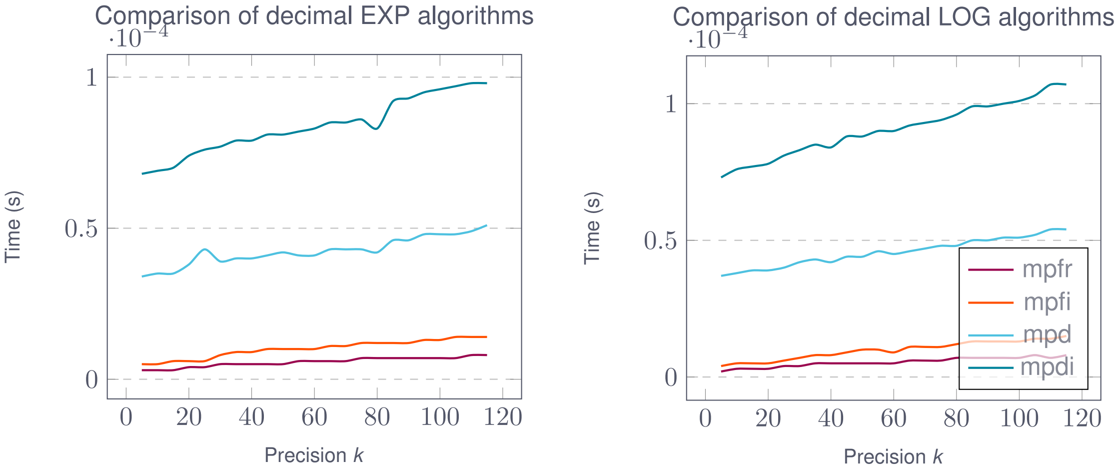
ADD and MUL execution time



EXP and LOG execution time



Conclusion and future work

Conclusion

- Development of two libraries, MPD and MPDI of a correctly rounded, reliable and fast decimal multiple precision arithmetic.
- Proof of the conversion algorithm
- Implementation of the basic mathematical functions, and the exponential and logarithm

Perspectives

- Add other functions: trigonometric functions (cos, sin, tan...)
- Proof of the other algorithms
- Expand the set of tests
- Source code available on demand

Questions

Thank you!