Enclosures of Roundoff Errors using SDP

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Errors and Proofs

GUARANTEED OPTIMIZATION

Input : Linear problem (LP), geometric, semidefinite (SDP)

Output : solution + **certificate** numeric-symbolic $\rightsquigarrow$ formal

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Reliable software/hardware embedded codes

Aerospace control

molecular biology, robotics, code synthesis, ...

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Efficient Verification of Nonlinear Systems

- Automated precision tuning of systems/programs
  analysis/synthesis
- Efficiency sparsity correlation patterns
- Certified approximation algorithms

Roundoff Error Bounds

Real : $f(\mathbf{x}) := x_1 \times x_2 + x_3$

Floating-point : $\hat{f}(\mathbf{x}, \mathbf{e}) := [x_1 x_2 (1 + e_1) + x_3](1 + e_2)$

Input variable constraints $\mathbf{x} \in \mathbf{X}$
Finite precision $\rightsquigarrow$ bounds over $\mathbf{e} \in \mathbf{E}$: $\mid e_i \mid \leqslant 2^{-53}$ (double)
Guarantees on absolute round-off error $\mid \hat{f} - f \mid$?

$\downarrow$ **Upper** Bounds $\downarrow$

max $\hat{f} - f$ ——— max $\hat{f} - f$

$\uparrow$ **Lower** Bounds $\uparrow$

$\downarrow$ **Lower** Bounds $\downarrow$

min $\hat{f} - f$ ——— min $\hat{f} - f$

$\uparrow$ **Upper** Bounds $\uparrow$

Nonlinear Programs

- Polynomials programs : $+, -, \times$

$$x_2 x_5 + x_3 x_6 + x_1(-x_1 + x_2 + x_3 - x_4 + x_5 + x_6)$$

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- Semialgebraic programs: $|\cdot|, \sqrt{}, /, \sup, \inf$

$$\frac{4x}{1 + \frac{x}{1.11}}$$

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$$\frac{4x}{1 + \frac{x}{1.11}}$$

- Transcendental programs: $\arctan, \exp, \log, \ldots$

$$\log(1 + \exp(x))$$

Existing Frameworks

Classical methods:

- Abstract domains [Goubault-Putot 11]

  FLUCTUAT: intervals, octagons, zonotopes

- Interval arithmetic [Daumas-Melquiond 10]

  GAPPA: interface with COQ proof assistant

Existing Frameworks

Recent progress:

- Affine arithmetic + SMT [Darulova 14], [Darulova 17]

  `rosa`: sound compiler for reals (SCALA)

- Symbolic Taylor expansions [Solovyev 15], [Chiang 17]

  `FPTaylor`: certified optimization (OCAML/HOL-LIGHT)

- Guided random testing `s3fp` [Chiang 14]

Contributions

Maximal Roundoff error of the program implementation of f:

$$r^\star := \max |\hat{f}(\mathbf{x}, \mathbf{e}) - f(\mathbf{x})|$$

Decomposition: **linear** term l w.r.t. $\mathbf{e}$ + nonlinear term h

$$\max |l| + \max |h| \geqslant r^\star \geqslant \max |l| - \max |h|$$

- Coarse bound of h with interval arithmetic
- **Semidefinite programming** (SDP) bounds for l:

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Sparse SDP relaxations	Robust SDP relaxations
$\downarrow$ **Upper** Bounds $\downarrow$	
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$\uparrow$ **Upper** Bounds $\uparrow$	

Sparse SDP relaxations **Robust SDP relaxations**

Contributions

1. General **SDP** framework for upper and lower bounds
2. Comparison with SMT & affine/Taylor arithmetic:
   $\rightsquigarrow$ **Efficient** optimization $\bigoplus$ **Tight** upper bounds
3. Extensions to transcendental/conditional programs
4. Formal verification of SDP bounds
5. Open source tools:

`Real2Float` (in OCaml and Coq)

↓ **Upper** Bounds ↓

↑ **Upper** Bounds ↑

`FPSDP` (in Matlab)

↑ **Lower** Bounds ↑
↓ **Lower** Bounds ↓

Introduction

Semidefinite Programming for Polynomial Optimization

Upper Bounds with Sparse SDP

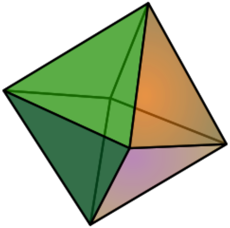
Lower Bounds with Robust SDP

Conclusion

What is Semidefinite Programming?

- Linear Programming (LP):

$$\min_{\mathbf{z}} \quad \mathbf{c}^\top \mathbf{z}$$
$$\text{s.t.} \quad \mathbf{A}\,\mathbf{z} \geqslant \mathbf{d} \ .$$



Polyhedron

  - Linear cost $\mathbf{c}$
  - Linear inequalities "$\sum_i A_{ij}\, z_j \geqslant d_i$"

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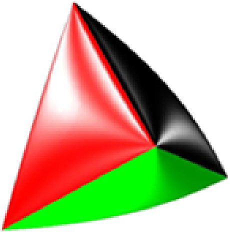

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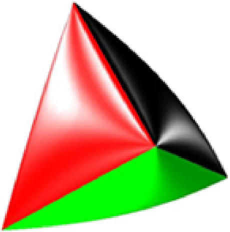

Spectrahedron

SDP for Polynomial Optimization

- Prove **polynomial inequalities** with SDP:

$$f(a,b) := a^2 - 2ab + b^2 \geqslant 0 \ .$$

- Find $\mathbf{z}$ s.t. $f(a,b) = \begin{pmatrix} a & b \end{pmatrix} \underbrace{\begin{pmatrix} z_1 & z_2 \\ z_2 & z_3 \end{pmatrix}}_{\succcurlyeq 0} \begin{pmatrix} a \\ b \end{pmatrix}$.

- Find $\mathbf{z}$ s.t. $a^2 - 2ab + b^2 = z_1 a^2 + 2 z_2 ab + z_3 b^2 \qquad (\mathbf{A}\,\mathbf{z} = \mathbf{d})$

- $\begin{pmatrix} z_1 & z_2 \\ z_2 & z_3 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}_{\mathbf{F}_1} z_1 + \underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{\mathbf{F}_2} z_2 + \underbrace{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}}_{\mathbf{F}_3} z_3 \succcurlyeq \underbrace{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}}_{\mathbf{F}_0}$

SDP for Polynomial Optimization

- Choose a cost $\mathbf{c}$ e.g. $(1, 0, 1)$ and solve:

$$\begin{aligned} \min_{\mathbf{z}} \quad & \mathbf{c}^\top \mathbf{z} \\ \text{s.t.} \quad & \sum_i \mathbf{F}_i z_i \succcurlyeq \mathbf{F}_0 \ , \quad \mathbf{A}\,\mathbf{z} = \mathbf{d} \ . \end{aligned}$$

- Solution $\begin{pmatrix} z_1 & z_2 \\ z_2 & z_3 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \succcurlyeq 0$ (eigenvalues 0 and 2)

- $a^2 - 2ab + b^2 = \begin{pmatrix} a & b \end{pmatrix} \underbrace{\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}}_{\succcurlyeq 0} \begin{pmatrix} a \\ b \end{pmatrix} = (a - b)^2 \, .$

- Solving **SDP** $\implies$ Finding **SUMS OF SQUARES** certificates

SDP for Polynomial Optimization

NP hard General Problem: $f^* := \min_{\mathbf{x} \in \mathbf{X}} f(\mathbf{x})$

- Semialgebraic set $\mathbf{X} := \{\mathbf{x} \in \mathbb{R}^n : g_1(\mathbf{x}) \geqslant 0, \ldots, g_m(\mathbf{x}) \geqslant 0\}$

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$$\overbrace{x_1 x_2}^{f} + \frac{1}{8} = \overbrace{\frac{1}{2}\left(x_1 + x_2 - \frac{1}{2}\right)^2}^{\sigma_0} + \overbrace{\frac{1}{2}}^{\sigma_1} \overbrace{x_1(1 - x_1)}^{g_1} + \overbrace{\frac{1}{2}}^{\sigma_2} \overbrace{x_2(1 - x_2)}^{g_2}$$

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- Bounded degree:
  $\mathcal{Q}_k(\mathbf{X}) := \left\{ \sigma_0 + \sum_{j=1}^m \sigma_j g_j, \text{ with } \deg \sigma_j \, g_j \leqslant 2k \right\}$

SDP for Polynomial Optimization

- **Hierarchy of SDP relaxations**:
  $\lambda_k := \sup_{\lambda} \left\{ \lambda : f - \lambda \in \mathcal{Q}_k(\mathbf{X}) \right\}$

- Convergence guarantees $\lambda_k \uparrow f^*$ [Lasserre 01]

- Can be computed with SDP solvers (CSDP, SDPA)

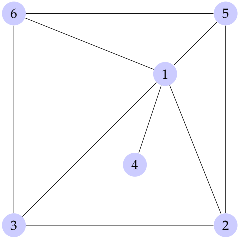
- **"No Free Lunch" Rule**: $\binom{n+2k}{n}$ SDP variables

- Extension to semialgebraic functions
  $\rightsquigarrow$ $r(\mathbf{x}) = p(\mathbf{x}) / \sqrt{q(\mathbf{x})}$ [Lasserre-Putinar 10]

Sparse SDP Optimization [Waki, Lasserre 06]

- Correlative sparsity pattern (csp) of variables

$$x_2x_5 + x_3x_6 - x_2x_3 - x_5x_6 + x_1(-x_1 + x_2 + x_3 - x_4 + x_5 + x_6)$$



1. Maximal cliques $C_1, \ldots, C_l$
2. Average size $\kappa \rightsquigarrow \binom{\kappa+2k}{\kappa}$ variables

$C_1 := \{1, 4\}$
$C_2 := \{1, 2, 3, 5\}$
$C_3 := \{1, 3, 5, 6\}$
Dense SDP: 210 variables
Sparse SDP: 115 variables

Polynomial Programs

Input: exact $f(\mathbf{x})$, approx $\hat{f}(\mathbf{x}, \mathbf{e})$, $\mathbf{x} \in \mathbf{X}$, $| e_i | \leqslant 2^{-53}$
Output: Bound for $f - \hat{f}$

1: Error $r(\mathbf{x}, \mathbf{e}) := f(\mathbf{x}) - \hat{f}(\mathbf{x}, \mathbf{e}) = \sum_{\boldsymbol{\alpha}} r_{\boldsymbol{\alpha}}(\mathbf{e}) \mathbf{x}^{\boldsymbol{\alpha}}$

2: Decompose $r(\mathbf{x}, \mathbf{e}) = l(\mathbf{x}, \mathbf{e}) + h(\mathbf{x}, \mathbf{e})$

3: $l(\mathbf{x}, \mathbf{e}) = \sum_{i=1}^{m} s_i(\mathbf{x}) e_i$

4: Maximal cliques correspond to $\{\mathbf{x}, e_1\}, \ldots, \{\mathbf{x}, e_m\}$

Dense relaxation: $\binom{n+m+2k}{n+m}$ SDP variables
Sparse relaxation: $m\binom{n+1+2k}{n+1}$ SDP variables

Preliminary Comparisons

$$f(\mathbf{x}) := x_2x_5 + x_3x_6 - x_2x_3 - x_5x_6 + x_1(-x_1 + x_2 + x_3 - x_4 + x_5 + x_6)$$

$$\mathbf{x} \in [4.00, 6.36]^6 , \quad \mathbf{e} \in [-\epsilon, \epsilon]^{15} , \quad \epsilon = 2^{-53}$$

- Dense SDP: $\binom{6+15+4}{6+15} = 12650$ variables $\rightsquigarrow$ **Out of memory**

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- Symbolic Taylor `FPTaylor` tool: 721ϵ ($21 \times$ more CPU)

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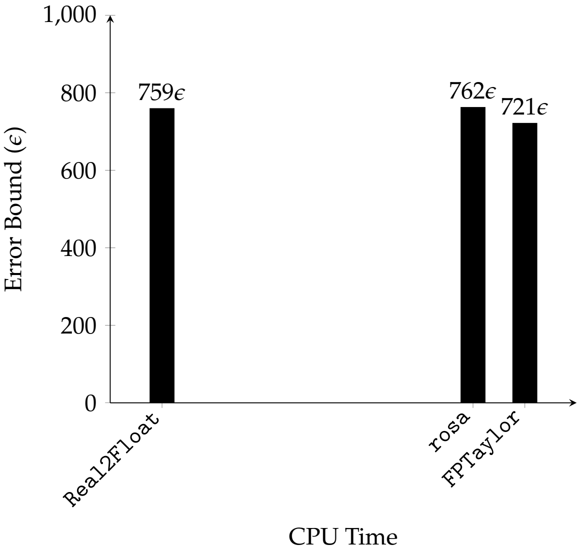
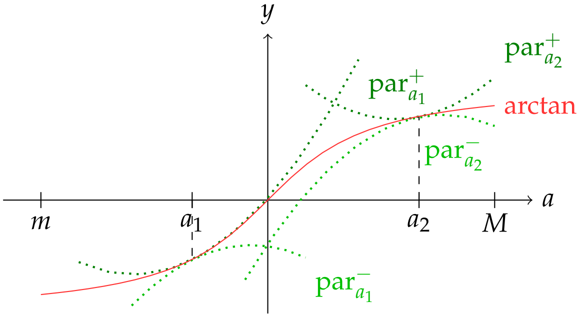
- Interval arithmetic: 922ϵ ($10 \times$ less CPU)
- Symbolic Taylor `FPTaylor` tool: 721ϵ ($21 \times$ more CPU)
- SMT-based `rosa` tool: 762ϵ ($19 \times$ more CPU)

Preliminary Comparisons



Extensions: Transcendental Programs

Reduce $f^* := \inf_{\mathbf{x} \in \mathbf{K}} f(\mathbf{x})$ to semialgebraic optimization



Extensions: Conditionals

```
if (p(x) ⩽ 0) f(x); else g(x);
```

DIVERGENCE PATH ERROR:

$$r^\star := \max\{ \max_{p(\mathbf{x}) \leqslant 0, p(\mathbf{x},\mathbf{e}) \geqslant 0} | \hat{f}(\mathbf{x},\mathbf{e}) - g(\mathbf{x}) |$$

$$\max_{p(\mathbf{x}) \geqslant 0, p(\mathbf{x},\mathbf{e}) \leqslant 0} | \hat{g}(\mathbf{x},\mathbf{e}) - f(\mathbf{x}) |$$

$$\max_{p(\mathbf{x}) \geqslant 0, p(\mathbf{x},\mathbf{e}) \geqslant 0} | \hat{f}(\mathbf{x},\mathbf{e}) - f(\mathbf{x}) |$$

$$\max_{p(\mathbf{x}) \leqslant 0, p(\mathbf{x},\mathbf{e}) \leqslant 0} | \hat{g}(\mathbf{x},\mathbf{e}) - g(\mathbf{x}) | \}$$

Method 1: `geneig` [Lasserre 11]

Generalized eigenvalue problem:

$$f^* := \min_{\mathbf{x} \in \mathbf{X}} f(\mathbf{x}) \leqslant \lambda_k := \sup_{\lambda} \quad \lambda$$
$$\text{s.t.} \quad \mathbf{M}_k(f\, \mathbf{y}) \succcurlyeq \lambda \mathbf{M}_k(\mathbf{y}).$$

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Uniform distribution moments: $\mathbf{y}_\alpha := \int_{\mathbf{X}} \mathbf{x}^\alpha d\mathbf{x}$

Localizing matrices $\mathbf{M}_k(f\ \mathbf{y})$:

$$\mathbf{M}_1(f\ \mathbf{y}) = \begin{array}{c} \\ 1 \\ x_1 \\ x_2 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & x_1 & x_2 \end{array} \\ \begin{pmatrix} \int_{\mathbf{X}} f(\mathbf{x}) d\mathbf{x} & \int_{\mathbf{X}} f(\mathbf{x}) x_1 d\mathbf{x} & \int_{\mathbf{X}} f(\mathbf{x}) x_2 d\mathbf{x} \\ \int_{\mathbf{X}} f(\mathbf{x}) x_1 d\mathbf{x} & \int_{\mathbf{X}} f(\mathbf{x}) x_1^2 d\mathbf{x} & \int_{\mathbf{X}} f(\mathbf{x}) x_1 x_2 d\mathbf{x} \\ \int_{\mathbf{X}} f(\mathbf{x}) x_2 d\mathbf{x} & \int_{\mathbf{X}} f(\mathbf{x}) x_2 x_1 d\mathbf{x} & \int_{\mathbf{X}} f(\mathbf{x}) x_2^2 d\mathbf{x} \end{pmatrix} \end{array}$$

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Theorem [Lasserre 11, de Klerk et al. 17]

$\lambda_k \downarrow f^*$ and $\lambda_k - f^* = \mathcal{O}(1/\sqrt{k})$

Method 2: `mvbeta` [DeKlerk et al. 17]

Elementary calculation with $f(\mathbf{x}) = \sum_\alpha f_\alpha \mathbf{x}^\alpha$:

$$f^* := \min_{\mathbf{x} \in \mathbf{X}} f(\mathbf{x}) \leqslant f_k^H := \min_{|\eta + \beta| \leqslant 2k} \sum_\alpha f_\alpha \frac{\gamma_{\eta+\alpha,\beta}}{\gamma_{\eta,\beta}}$$

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Multivariate beta distribution moments:

$$\gamma_{\eta,\beta} := \int_{\mathbf{X}} \mathbf{x}^{\eta} (1 - \mathbf{x})^{\beta} d\mathbf{x} \, .$$

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Theorem [DeKlerk et al. 17]

$f_k^H \downarrow f^*$ and $\lambda_k - f^* = \mathcal{O}\left(1/\sqrt{k}\right)$

Method 3: `robustsdp`

Robust SDP with $l(\mathbf{x}, \mathbf{e}) = \sum_{i=1}^{m} s_i(\mathbf{x}) e_i$:

$$l^* := \min_{(\mathbf{x},\mathbf{e}) \in \mathbf{X} \times \mathbf{E}} l(\mathbf{x}, \mathbf{e}) \leqslant \lambda'_k := \sup_{\lambda} \quad \lambda$$

$$\text{s.t.} \quad \forall \mathbf{e} \in \mathbf{E}, \ \mathbf{M}_k(l\, \mathbf{y}) \succcurlyeq \lambda \mathbf{M}_k(\mathbf{y}).$$

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Linearity $\rightsquigarrow \mathbf{M}_k(l \ \mathbf{y}) = \sum_{i=1}^{m} e_i \mathbf{M}_k(s_i \ \mathbf{y})$

Factorization $\rightsquigarrow \mathbf{M}_k(s_i \ \mathbf{y}) = \mathbf{L}_k^i \mathbf{R}_k^i$

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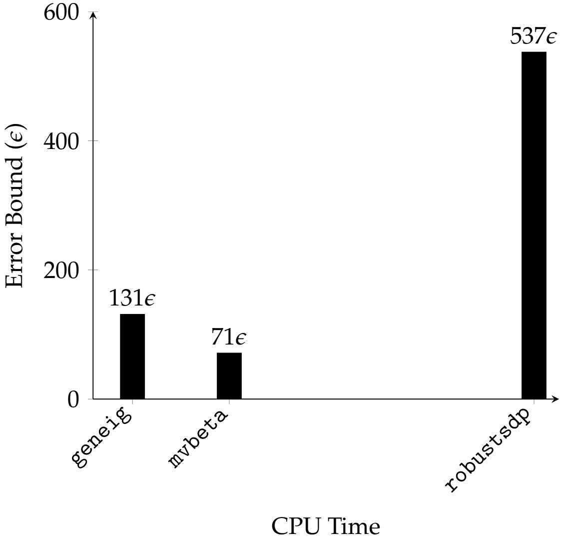
Theorem following from [El Ghaoui et al. 98]

$$\lambda'_k \downarrow l^* \text{ and } \lambda'_k = \sup_{\lambda, \mathbf{S}, \mathbf{G}} \quad \lambda$$
$$\text{s.t.} \quad \begin{pmatrix} -\lambda \mathbf{M}_k(\mathbf{y}) - \mathbf{L}_k\ \mathbf{S}\ \mathbf{L}_k{}^T & \mathbf{R}_k{}^T + \mathbf{L}_k\ \mathbf{G} \\ \mathbf{R}_k - \mathbf{G}\ \mathbf{L}_k{}^T & \mathbf{S} \end{pmatrix} \succcurlyeq 0,$$
$$\mathbf{S}^T = \mathbf{S}, \mathbf{G}^T = -\mathbf{G}.$$

Benchmark `kepler0` with $k = 2$

↑ **Lower** Bounds ↑
↓ **Lower** Bounds ↓



Conclusion

Sparse/Robust SDP relaxations for NONLINEAR PROGRAMS:

- Polynomial and transcendental programs

- Certified $\rightsquigarrow$ Formal roundoff error bounds
(Joint work with T. Weisser and B. Werner)

- `Real2Float` and `FPSDP` open source tools:

  `http://nl-certify.forge.ocamlcore.org/real2float.html`

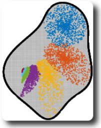
  `https://github.com/magronv/FPSDP`

Conclusion

Further research:

- Automatic **FPGA** code generation

- Handling `while`/`for` loops



Master / PhD Positions Available !

End

Thank you for your attention!

`http://www-verimag.imag.fr/~magron`

- V. Magron, G. Constantinides, A. Donaldson. Certified Roundoff Error Bounds Using Semidefinite Programming, *ACM Trans. Math. Softw*, 2017.

↓ **Upper** Bounds ↓

↑ **Upper** Bounds ↑

- V. Magron. Interval Enclosures of Upper Bounds of Roundoff Errors using Semidefinite Programming, `arxiv.org/abs/1611.01318`.

↑ **Lower** Bounds ↑
↓ **Lower** Bounds ↓